MATH410 - 0401

Ash Dorsey
ash@ash.lgbt

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Contents

1. Preliminaries

1.1. Set Operations

Let A and B be sets.

1.1.1. Intersection

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

1.1.2. Union

$A \cup B = \{x \mid x \in A \vee x \in B\}$

1.2. Singleton

A set containing only one element

1.3. Universal Set

Denoted U, all sets are subsets of U.

1.4. Properties of union and intersection

1. $A \cap B \subset A$
2. $A \cap B \subset B$
3. $A \subset A \cup B$
4. $B \subset A \cup B$

1.5. Complement

If there is a universal set, A^C is the set of all elements in the universal set but not in A.

1.6. Minus

Denoted $A \setminus B$, it is all the elements in A that are not in the elements in B.

Furthermore, $A \setminus B = A \cap B^C$.

1.7. Subset

A set A is a subset of a set B if every element in A is also in B.

For example, if $A = \{1, 2, 3\}, B = \{a, b, c, 1, 2, 3\}$, $A \subset B$.

1.8. Empty Set

The empty set has nothing in it. It is denoted $\varnothing$.

1.9. Functions

A function from a set A to a set B associates an element of B with each element of A.

A is called the domain, and B is called the codomain.

$$f : A \to B$$
$$f(x) = 2x + 3$$

If $A = \{1, 2, 4\}$, $\{5, 7, 11\} \subset B$.

For each element $x \in A$, f associates x with someone element $y \in B$.

We write $f(x) = y$. We say f maps x to y.

1.9.1. Example

$$f(x) = \frac{1}{x-2}$$
$$f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$$

1.10. Image

Let there be a function $f : A \to B$

$$f(A) = \{y \mid y = f(x) \land x \in A\}$$

The image of A is then $f(A)$

1.11. Injective/One-to-one

$f : A \to B$ is injective if:

$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \to x_1 = x_2$

Therefore, if $f(x) = y$, then x is the only element of A that f maps to y.

1.11.1. Example

$f(x) = 3x + 1$ is one-to-one, $f(x) = x^2$, is not one-to-one

1.12. Surjective/Onto

$f : A \to B$ is a surjection if $f(A) = B$.

$\forall x \in B, \exists y \in A, x = f(y)$

1.13. Bijective

If f is injective and surjective, then f is bijective.

1.14. Set Cardinality

A set's cardinality is the number of elements in the set.

Two sets A and B have the same cardinality if there exists a bijection function $f : A \to B$.

1.14.1. Example

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{A}$ have the same cardinality, called countably infinite.

1.15. Existence of Inverses

An inverse of a function exists iff a function is bijective. The inverse of f is denoted f^{-1}.

1.16. Axioms of Real Numbers

$\mathbb{R}$ are built based on three axioms:

1. Field Axioms
    1. Commutativity of addition: $\forall a, b \in \mathbb{R}, a + b = b + a$
    2. Associativity of addition: $\forall a, b, c \in \mathbb{R}, a + (b + c) = (a + b) + c$
    3. Additive identity: there is a real number, denoted 0, such that $\forall a \in \mathbb{R}, 0 + a = a + 0 = a$
    4. Additive inverse: for each $a \in \mathbb{R}, \exists b \in \mathbb{R}$ so that $a + b = 0$.

5. Commutativity of multiplication: $\forall a, b \in \mathbb{R}, a \times b = b \times a$
6. Associativity of multiplication: $\forall a, b, c \in \mathbb{R}, a \times (b \times c) = (a \times b) \times c$
7. Multiplicative identity: there is a real number, denoted 1, such that $\forall a \in \mathbb{R}, 1 \times a = a \times 1 = a$
8. Multiplicative inverse: $\forall a \in \mathbb{R}, a \neq 0 \to \exists b \in \mathbb{R}, ab = 1$
9. Distributive property: $\forall a, b, c \in \mathbb{R}, a(b + c) = ab + ac$.
10. Nontriviality, $0 \neq 1$.

2. Positivity axiom
   - How Reals are ordered
3. Completeness axiom
   - Reals have no gaps

1.16.1. Proposition 1

The element 0 is the only real number satisfying the additive identity property.

1.16.1.1. Proof

Suppose $\exists z \in \mathbb{R}$, so that $\forall a \in \mathbb{R}, z + a = a$.

Let b be the additive inverse of a.

$$\begin{aligned} & z \\ & = z + 0 \\ & = z + (a + b) \\ & = (z + a) + b \\ & = a + b \\ & = 0 \end{aligned}$$

Therefore, $z = 0$.

1.16.2. Proposition 2

The element 1 is the only real number satisfying the multiplicative identity property.

1.16.2.1. Proof

Suppose $\exists z \in \mathbb{R}$, so that $\forall a \in \mathbb{R}, z \times a = a$.

Assume $a \neq 0$, and then let b be the multiplicative inverse of a. Then,

$$\begin{aligned} & z \\ & = 1 \times z \\ & = a \times b \times z \\ & = z \times a \times b \\ & = a \times b \\ & = 1 \end{aligned}$$

1.16.3. Proposition 3

$\forall a \in \mathbb{R}, 0a = a0 = 0$

1.16.3.1. Proof

Let $a \in \mathbb{R}$, and let b be the additive inverse of a.

$$
\begin{aligned}
&0 \\
&= a + b \\
&= 1a + b \\
&= (1 + 0)a + b \\
&= 1a + 0a + b \\
&= a + 0a + b \\
&= a + b + 0a \\
&= 0 + 0a \\
&= 0a \\
&= a0
\end{aligned}
$$

1.16.4. Proposition 4

If $a, b \in \mathbb{R}$ such that $ab = 0$, $a = 0 \vee b = 0$

1.16.4.1. Proof

Let $a, b \in \mathbb{R}$, such that $ab = 0$.

If $a = 0$, $a = 0 \vee b = 0$ is true, because $a = 0$.

Otherwise, WLOG, suppose $a \neq 0$.

Then, there is a multiplicative inverse c of a such that $ca = 1$.

$$
\begin{aligned}
&0 \\
&= c0 \\
&= c(ab) \\
&= (ca)b \\
&= 1b \\
&= b
\end{aligned}
$$

Therefore $b = 0$, and similarily $b \neq 0 \to a = 0$, and so $a = 0 \vee b = 0$ is always true.

1.16.5. Proposition 5

$\forall a \in \mathbb{R}$ there is a unique solution x to the equation $a + x = 0$.

1.16.5.1. Proof

Let $a \in \mathbb{R}$.

Let b be the additive inverse of a. Therefore, $x = b$ is a solution.

By contradiction, suppose $x = y$ also a solution to $a + x = 0$, $b \neq y$.

$$
\begin{aligned}
&b \\
&= b + 0 \\
&= b + (a + y) \\
&= (b + a) + y \\
&= (a + b) + y \\
&= 0 + y
\end{aligned}
$$

$$= y$$

But by assumption $b \neq y$, so our assumption was wrong, and therefore there is only one solution x to $a + x = 0$.

Since each $a \in \mathbb{R}$ has a unique additive inverse, we denote it by $-a$ and define subtraction by $a - b = a + (-b)$

A similar proof show for any nonzero $a \in \mathbb{R}$, we have a unique multplicative inverse, denoted by a^{-1}, and define the quotient of a and b as

$$\frac{a}{b} = ab^{-1}$$

1.17. Exercises

Prove the following:

1. $-(-a) = a$
2. $\left(b^{-1}\right)^{-1} = b$
3. $(-b)^{-1} = -\left(b^{-1}\right)$
4. $(-a)b = -(ab)$
5. $\left(ab^{-1}\right)^{-1} = a^{-1}b$

1.18. Positivity Axioms of Real Numbers

There is a subset $\mathbb{P}$, called positive numbers, of real numbers with the following properties:

1. $a, b \in \mathbb{P} \to ab, a + b \in \mathbb{P}$
2. $\forall a \in \mathbb{R}$, exactly one of the following is true:
   1. $a \in \mathbb{P}$
   2. $-a \in \mathbb{P}$
   3. $a = 0$

These axioms let us define $>$ or $<$ operators.

Let $a, b \in \mathbb{R}$.

1. $a > b$ if $a - b \in \mathbb{P}$
2. $a < b$ if $b - a \in \mathbb{P}$
3. $a = b$ if $a - b = 0$

Then, $a \in \mathbb{P} \leftrightarrow a > 0$.

1.18.1. Proposition

If $a \in \mathbb{R} \setminus \{0\}$, then $a^2 > 0$.

1.18.2. Proposition

If $a \in \mathbb{P}$, then $a^{-1} \in \mathbb{P}$

1.18.3. Proposition

If $a > b$, $(c > 0 \to ac > bc) \land (c < 0 \to ac < bc)$.

1.19. Interval Notation

Let $a < b$, then we define

1. $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
2. $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
3. $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
4. $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

1.20. Inductive Set

A set S is inductive iff

1. $1 \in S$
2. $\forall x \in S, x + 1 \in S$

1.21. Natural Numbers: $\mathbb{N}$

Let I be the collection of all inductive sets.

Define:

$$\mathbb{N} = \bigcap_{S \in I} S$$

1.21.1. Proposition

The natural numbers $\mathbb{N}$ are inductive

1.21.1.1. Proof

First point:

$$\forall S \in I, 1 \in S \rightarrow 1 \in \bigcap_{S \in I} S$$

Second property:

Let $x \in \mathbb{N}$. Then $\forall S \in I, x \in S$ and $\forall S \in I, x + 1 \in \mathbb{S}$

Therefore, $\mathbb{N}$ is inductive.

1.22. Properties of $\mathbb{N}$

For $n, m \in \mathbb{N}$:

1. $m + n \in \mathbb{N}$
2. $m \times n \in \mathbb{N}$

1.23. Integers: $\mathbb{Z}$

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-x \mid x \in \mathbb{N}\}$$

1.24. Properties of $\mathbb{Z}$

For $n, m \in \mathbb{Z}$:

1. $m + n \in \mathbb{Z}$
2. $m - n \in \mathbb{Z}$
3. $m \times n \in \mathbb{Z}$

1.25. Rationals: $\mathbb{Q}$

$$\mathbb{Q} = \left\{ \frac{n}{m} \,\middle|\, n, m \in \mathbb{Z}, m \neq 0 \right\}$$

1.25.1. Proposition

$\mathbb{Q}$ satisfies the field and positivity axioms.

1.25.2. Proposition

$$\forall a, b \in \mathbb{Q}, a < b, \exists c \in \mathbb{Q}, a < c < b$$

1.25.2.1. Proof:

Let $c = \frac{a+b}{2}$. $c \in \mathbb{Q}$ by the field axioms.

$$c - a = \frac{a+b}{2} - a = \frac{b-a}{2} = \frac{1}{2}(b-a) > 0$$

1.25.3. Facts

1. Each rational number can be written as $\frac{m}{n}$ where either $2 \nmid m \vee 2 \nmid n.$.
2. An integer n is even iff n^2 is even.

1.25.4. Formational Example

$$f(x) = x^2 - 2$$

$f(1) < 0, f(2) > 0$ It would be good if $\exists c, 1 < c < 2, f(c) = 0$.

But this is impossible in the rationals.

1.25.5. Proposition

There is no rational number a so that $a^2 = 2$.

1.25.5.1. Proof

On the contrary, assume $a \in \mathbb{Q}$ where $a^2 = 2$.

Therefore, $a = \frac{m}{n}, m, n \in \mathbb{Z}$. By a fact we know that either m or n is odd.

$$a = \frac{m}{n}$$
$$2 = a^2 = \frac{m^2}{n^2}$$
$$2n^2 = m^2$$

Therefore, m^2 is even, which means that m is even.

Since m is even, $\exists k \in \mathbb{Z}, m = 2k$.

Thus,

$$2n^2 = m^2$$
$$2n^2 = 4k^2$$
$$n^2 = 2k^2$$

Therefore n^2 is even, which means that n is even.

Both n and m are even, which is a contradiction.

Therefore, there is no $a \in \mathbb{Q}$ with $a^2 = 2$.

2. Tools of Analysis

2.1. Bounded Above

A nonempty $S \subset \mathbb{R}$ is bounded above if

$$\exists c \in \mathbb{R}, \forall x \in S, x \leq c$$

c is then an upper bound.

2.2. Bounded Below

A nonempty $S \subset \mathbb{R}$ is bounded below if

$$\exists c \in \mathbb{R}, \forall x \in S, c \leq x$$

c is then a lower bound.

2.3. Completeness Axiom

Every $S \subset \mathbb{R}$ that is bounded above has a least upper bound c so that $\forall x \in S, x \leq c$ and if b is an upper bound of S, $c \leq b$.

2.4. Definition of Real Numbers

Any set that satisfies the

- Field axioms
- Positivity Axoims
- Completeness axiom

is equivalent to $\mathbb{R}$

2.5. Supremum

The supremum of a set is the least upper bound of that set.

2.5.1. Example

$\sup([2, 3)) = 3$.

2.6. Infimum

Every $S \subset \mathbb{R}$ bounded below has a greatest lower bound, $c = \inf S$

2.7. $\sqrt{2} \in \mathbb{R}$

IOW, $\exists a \in \mathbb{R}^{\geq 0}, a^2 = 2$

2.7.1. Proof

Let $S = \{x \in \mathbb{R} \,|\, x \geq 0, x^2 \leq 2\}$.

2 is an upper bound. Let $a = \sup(S)$

On the contrary, assume $a^2 > 2$. Let $r = \frac{a^2-2}{2a}$. $r > 0$.

$$(a-r)^2 = a^2 - 2ar + r^2 \overset{r>0}{>} a^2 - 2ar = a^2 - \cancel{2a}\frac{a^2-2}{\cancel{2a}} = 2$$

$$\forall x \in S, (a-r)^2 > 2, a - r > x$$

$$a - r < a, a \leq 2.$$

(incomplete)

2.8. Archimedean Property

1. For all $c \in \mathbb{R}, c > 0$, there exists $n \in \mathbb{N}$ with $n > c$
2. For all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$

(these statements are equivalent.)

2.8.1. Proof

We will prove the first statement.

On the contrary, assume $\exists c \in \mathbb{R}, \forall n \in \mathbb{N}, n \leq c$.

Let $b = \sup \mathbb{N}$. This must exist because c is an upper bound of $\mathbb{N}$. Since b is the least upper bound, $b - \frac{1}{2}$ is not an upper bound. Therefore, $\exists n \in \mathbb{N}, n > b - \frac{1}{2}$.

Then, $n + 1 > \left(b - \frac{1}{2}\right) + 1 = b + \frac{1}{2} > b$. Therefore, $n + 1 > b$, but $n + 1 \in \mathbb{N}$. Therefore, c does not exist, and so the first statement is true.

2.9. Integers will not exist between integers

Let $n \in \mathbb{Z}$. There is no integer in $(n, n + 1)$.

2.9.1. Proof

Consider the set $\{k \mid k \in \mathbb{N}, k \geq 1\}$. It is an inductive subset of $\mathbb{N}$ (by the positivity axioms), and therefore it is $\mathbb{N}$. Therefore, $\forall k \in \mathbb{N}, k \geq 1$.

Since all positive integers are in $\mathbb{N}$, the interval $(0, 1) \cap \mathbb{N} = \varnothing$.

Now suppose $k \in (n, n + 1) \cap \mathbb{Z}$. Then, $n < k < n + 1$. Therefore, $0 < k - n < 1$. $k - n \in \mathbb{Z}$. Since $k - n > 0$ and $k \in \mathbb{Z}$, $k \in \mathbb{N}$. Therefore, $k - n \in (0, 1) \cap \mathbb{N}$, but $(0, 1) \cap \mathbb{N} = \varnothing$, so k does not exist, and there is a contradiction.

2.10. Sets of integers have maxima

If S is a nonempty set of integers that is bounded above, then S has a maximum.

2.10.1. Proof

Let $a = \sup S$. a is the least upper bound of S. Therefore, $a - 1$ is not an upper bound. Therefore, $\exists n \in S, a - 1 < n$. Therefore, $a < n + 1$. Then, $S \subset (-\infty, n + 1)$. By the previous result, $(n, n + 1)$ contains no elements.

$(-\infty, n + 1) = (-\infty, n] \cup (n, n + 1)$. Therefore, $S \subset (-\infty, n]$, and $n \in S$, so n is the maximum of S.

2.11. One integer exist in each interval of size 1

For any $c \in \mathbb{R}$, $\exists! n \in \mathbb{N}, n \in [c, c + 1)$

2.11.1. Proof

Let $S = \{n \mid n \in \mathbb{Z}, n < c + 1\}$.

If $c \geq 0$, then $0 \in S$.

If $c < 0$, by the Archimedean property, $\exists m \in \mathbb{N}, m > -c$. Thus, $-m < c < c + 1$, therefore $-m \in S$.

Therefore $S \neq \varnothing$.

By the previous result, S has a maximum n.

By defintion of S, $n < c + 1$.

If $n < c \to n + 1 < c + 1$ and therefore $n + 1 \in S$. But this is impossible since n was the max of S and therefore $n \geq c$.

Therefore, $c \leq n < c + 1$ and so $n \in [c, c + 1)$.

Let $n, m \in \mathbb{Z} \cap [c, c + 1)$.

WLOG, assume $m \leq n$:

$$0 \leq n - m < (c + 1) - c = 1$$
$$0 \leq n - m < 1$$

So $n - m \in [0, 1) \cap \mathbb{Z}$ and therefore $n - m = 0 \to n = m$.

2.12. A rational exists between any two reals

For any $a, b \in \mathbb{R}$ with $a < b, \exists c \in \mathbb{Q}$ with $a < c < b$, and therefore $c \in (a, b)$.

2.12.1. Proof

Let $\frac{1}{n} < b - a$. Then: (This is incomplete)

$$nb - 1 \leq m \leq nb$$
$$a < b - \frac{1}{n} \leq \frac{m}{n} < b$$

2.13. Dense

A set $S \subset \mathbb{R}$ is dense iff $\forall a, b \in \mathbb{R}, a < b \to S \cap (a, b) \neq \varnothing$.

2.13.1. Examples

$\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{A}, \mathbb{Q} \setminus \mathbb{Z}$

2.14. Absolute Value

$$|\cdot| : \mathbb{R} \to \{x \in \mathbb{R} \,|\, x \geq 0\}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2.14.1. Properties

1. If $d \geq 0$, $|c| \leq d$ iff $d \leq c \leq d$.
2. $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$

2.15. Triangle Inequality

$$\forall a, b, \in \mathbb{R}, |a + b| \leq |a| + |b|$$

2.15.1. Proof

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

$$\leftrightarrow |a + b| \leq |a| + |b|$$

2.16. Some Sums

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

$$\forall r \neq 0, \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

3. Sequences

3.1. Sequences

A sequence is some function $a : \mathbb{N} \to \mathbb{R}$. This is typically written as some a_n rather than $a(n)$.

The entire sequence is denoted like $\{a_n\}$.[1]

3.1.1. Examples

$$\{n^2\} = \{1, 4, 9, 16, ...\}$$

$$\{1 + (-1)^n\} = \{0, 2, 0, 2, 0, 2, ...\}$$

$$\{a_n\} \text{ where } a_n \in \left(0, \frac{1}{n}\right)$$

$$a_1 = 1, a_{n+1} = 3a_n + 1, \{a_n\} = \{1, 4, 13, ...\}$$

$$\{a_n\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\right\}$$

$$\{a_n\} = \left\{\sum_{k=0}^{n} r^k\right\}$$

$$\{a_n\} = \left\{\sum_{k=1}^{n} \frac{1}{k}\right\}$$

3.2. Convergence of Sequence

A sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists N, n \geq N \to |a_n - a| < \varepsilon$.

Therefore, we say:

$$\lim_{n \to \infty} a_n = a$$

3.2.1. Example: Converge

Prove $\{a_n\} = \left\{\frac{1}{n}\right\}$ converges to $a = 0$.

Fix $\varepsilon > 0$.

Let $N > \frac{1}{\varepsilon}$. Therefore, $n \geq N > \frac{1}{\varepsilon}$ and:

$$n > \frac{1}{\varepsilon}$$

$$\frac{1}{n} < \varepsilon$$

$$a_n < \varepsilon$$

$$|a_n| < \varepsilon$$

$$|a_n - a| < \varepsilon$$

$$\therefore \square$$

[1]Or perhaps (a_n) or $(a_n)_{n=1}^{\infty}$

3.2.2. Example: Does not converge

Prove $\{a_n\} = \{1 + (-1)^n\}$ does not converge.

Assume $\{a_n\}$ converges to a. Let $\varepsilon = 1$, $a \in \mathbb{R}$, let $N > 0$. Let n_1 be the smallest even number larger than N and let n_2 be the smallest odd number larger than N.

$$|2 - a| = \left|a_{n_1} - a\right| < \varepsilon$$

$$|a| = |0 - a| = \left|a_{n_2} - a\right| < \varepsilon$$

$$\begin{aligned} 2 &= |2 - a + a| \\ &\leq |2 - a| + |a| \\ &< |2 - a| + \varepsilon \\ &< \varepsilon + \varepsilon \\ &< 2 \end{aligned}$$

But $2 < 2$ is false, so it does not converge.

3.3. Cannot converge to two different values

$$\left(\lim_{n \to \infty} a_n = a \land \lim_{n \to \infty} a_n = b\right) \to (a = b)$$

3.3.1. Proof

Fix $\varepsilon > 0$. Then $\exists N_1$ such that if $n \geq N$ then $|a_n - a| < \frac{\varepsilon}{2}$, and $\exists N_2$ such that $n \geq N_2$, then $|a_n - b| < \varepsilon$.

Consider $\varepsilon = b - a$. (WLOG, $b > a$).

Let $N > \max\{N_1, N_2\}$. Then if $n \geq N$:

$$\begin{aligned} \varepsilon &= |b - a| \\ &= |b - a_n + a_n - a| \\ &\leq |b - a_n| + |a_n - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

But $\varepsilon < \varepsilon$ is impossible, and so $b > a$ is false, and therefore $a \neq b$.

3.4. Create Limit to Zero

$$\{a_n\} \to a \leftrightarrow \{a_n - a\} \to 0$$

3.4.1. Proof

Prove $\to$ direction. Fix $\varepsilon > 0$. $\exists N > 0, n \geq N \to |a_n - a| < \varepsilon$.

Therefore, $|(a_n - a) - 0| < \varepsilon$.

Prove $\leftarrow$ direction. Fix $\varepsilon > 0$. $\exists N > 0, n \geq N \to |(a_n - a) - 0| < \varepsilon$.

Therefore, $|a_n - a| < \varepsilon$.

Therefore $\leftrightarrow$ is proven

3.5. All Convergent Sequences are Bounded

If $\{a_n\}$ converges, then $\exists M \geq 0, \forall n, |a_n| \leq M$

3.5.1. Proof

Let a be the limit of $\{a_n\}$.

Choose $\varepsilon = 1$.

$\exists N, n \geq N \to |a_n - a| < 1$.

Let $M = \max\{a + 1, |a_1|, |a_2|, ..., |a_{N-1}|\}$

Clearly, $\forall n \in \mathbb{N} \cap [1, N - 1], M \geq |a_n|$.

If $n \geq N$, then $M - a_n = M - a_n - a + a$.

Then

$$\begin{aligned} M - a_n &= M - a_n + a - a \\ &= (M - a) - (a_n - a) \\ &> M - a - 1 \\ &= M - (a + 1) \\ &\geq 0 \end{aligned}$$

3.6. Comparison Lemma

Suppose $\{a_n\} \to a$. Then $\{b_n\} \to b$ if $\exists c \geq 0, \exists N_1 > 0, \forall n \geq N_1, |b_n - b| < c|a_n - a|$.

3.6.1. Proof

Fix $\varepsilon > 0$.

$$\exists N_2, n \geq N_2 \to |a_n - a| < \frac{\varepsilon}{C}$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$:

$$|b_n - b| < C|a_n - a| < C\frac{\varepsilon}{C} = \varepsilon$$

3.7. Addition of Sequences

If $\{a_n\} \to a$ and $\{b_n\} \to b$, then $\{a_n + b_n\} \to a + b$.

3.7.1. Proof

Fix $\varepsilon > 0$.

$\exists N_1, N_2, n \geq N_1 \to |a_n - a| < \frac{\varepsilon}{2}, n \geq N_2 \to |b_n - b| < \frac{\varepsilon}{2}$.

Then, let $N = \max\{N_1, N_2\}$. Therefore:

$$n \geq N \to |a_n - a| < \frac{\varepsilon}{2} \land |b_n - b| < \frac{\varepsilon}{2}$$

$$n \geq N \to |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

3.8. Multiply Sequence by Constant

If $\{a_n\} \to a$ and $\alpha \in \mathbb{R}$, then $\{\alpha a_n\} \to \alpha a$.

3.8.1. Proof

Fix $\varepsilon > 0$. $\forall n, |\alpha a_n - \alpha a| = |\alpha||a_n - a| < 2|\alpha||a_n - a|$

Apply the comparison lemma.

3.9. Multiply Zero-Valued Sequence by Zero-Valued Sequence

If $\{a_n\} \to 0$ and $\{b_n\} \to 0$, then $\{a_n b_n\} \to 0$.

3.9.1. Proof

Fix $\varepsilon > 0$. $\exists N_1, N_2, n \geq N_1 \to |a_n - 0| < \sqrt{\varepsilon}, n \geq N_2 \to |b_n - 0| < \sqrt{\varepsilon}$.

Let $N = \max\{N_1, N_2\}$.

$$n \geq N \to |a_n b_n - 0| = |a_n b_n| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

3.10. Multiply Sequence by Sequence

If $\{a_n\} \to a$ and $\{b_n\} \to b$, then $\{a_n b_n\} \to ab$.

3.10.1. Proof

Let $\alpha_n = a_n - a$, $\beta_n = b_n - b$. Then $\{\alpha_n\} \to 0$, $\{\beta_n\} \to 0$.

Also, $|a_n b_n - ab| = |(\alpha_n + a)(\mathrm{B}_n + b) + ab| = |\alpha_n \beta_n + a\beta_n + b\alpha_n|$.

Observe that $\{\alpha_n \beta_n\} \to 0$, $\{a\beta_n\} \to 0$, $\{b\alpha_n\} \to 0$.

Let $c_n = \{\alpha_n \beta_n + a\beta_n + b\alpha_n\}$. Then, $\{c_n\} \to 0$ and so $\{a_n b_n - ab\} \to 0$, and so $\{a_n b_n\} \to ab$.

3.11. Reciprocal of sequence

If $\{b_n\} \to b$, and $b \neq 0$, then $\left\{\frac{1}{b_n}\right\} \to \frac{1}{b}$.

3.11.1. Proof

We must find $C, N_1 > 0$, so that if $n \geq N_1$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq C|b - b_n|$.

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n||b|}|b - b_n|$$

Therefore, we need to show that $\left\{\frac{1}{|b_n|}\right\}$ is bounded.

Observe that $|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n|$.

Therefore, $|b_n| \geq |b| - |b - b_n|$.

Let $\varepsilon = \left|\frac{b}{2}\right|$.

Then, $\exists N > 0$ such that if $n \geq N_1$, then $|b_n - b| < \varepsilon = \frac{|b|}{2}$

Then, $|b_n| \geq |b| - |b - b_n| \geq |b| - \frac{|b|}{2} = \frac{|b|}{2}$.

Then, $\frac{1}{|b_n|} \leq \frac{2}{|b|}$.

Therefore:

$$\left|\frac{1}{b_n}-\frac{1}{b}\right|=\frac{1}{|b_n||b|}|b_n-b|\leq\frac{2}{|b|^2}|b_n-b|$$

And therefore, by the comparison lemma, $\left\{\frac{1}{b_n}\right\}\to\frac{1}{b}$.

3.12. Division of Sequences

If $\{a_n\}\to a$, and $\{b_n\}\to b, b\neq 0$, then $\left\{\frac{a_n}{b_n}\right\}\to\frac{a}{b}$.

3.12.1. Proof

Let $c_n=\frac{1}{b_n}$. Then, $\left\{\frac{a_n}{b_n}\right\}=\{a_n c_n\}$.

By a proposition, $\{c_n\}\to\frac{1}{b}$, and so $\{a_n c_n\}\to\frac{a}{b}$ by a proposition.

4. Continuous Functions

4.1. A subset of the reals is dense based on sequential denseness

$S \subset \mathbb{R}$ is dense iff, $\forall x \in \mathbb{R}, \exists \{a_n\} \subset S, \{a_n\} \to x$.

4.1.1. Proof

Suppose $S \subset \mathbb{R}$ is dense.

Let $x \in \mathbb{R}$. Let $a_n \in S \cap \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$.

Fix $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$.

Then if $n \geq N$,

$$a_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$$
$$x - \frac{1}{n} < a_n < x + \frac{1}{n}$$
$$-\frac{1}{n} < a_n - x < \frac{1}{n}$$
$$|a_n - x| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

For the alternate direction, suppose that $\forall x \in \mathbb{R}, \exists \{a_n\} \subset S$, $\{a_n\} \to x$. Let $(a, b) \subset \mathbb{R}$. Let $x \in \frac{a+b}{2}$. Let $\varepsilon = \frac{b-a}{2}$.

$\exists \{a_n\} \to x, \exists N > 0, n \geq N \to |a_n - x| < \varepsilon$,

So:

$$-\varepsilon < a_n - x < \varepsilon$$
$$x - \varepsilon < a_n < x + \varepsilon$$
$$a_n \in (x - \varepsilon, x + \varepsilon)$$
$$a_n \in (a, b)$$

Since $a_n \in S$, and $a_n \in (a, b)$, $a_n \in S \cap (a, b)$, and so S is dense.

4.2. Nonnegative sequences converge to a nonnegative number

If $a_n \geq 0$ and $\{a_n\} \to a$, then $a \geq 0$.

4.2.1. Proof

Suppose $a < 0$ and let $\varepsilon = \frac{|a|}{2}$. Then $\exists N, n \geq N \to |a_n - a| < \varepsilon = \frac{|a|}{2}$, so

$$-\frac{|a|}{2} < a_n - a < \frac{|a|}{2}$$
$$\frac{a}{2} < a_n - a < -\frac{a}{2}$$
$$\frac{3a}{2} < a_n < \frac{a}{2} < 0$$
$$a_n < 0$$

But a_n was supposed to have the property $a_n \geq 0$. Therefore, the lemma holds.

4.3. Squeeze Theoremish Proposition

Suppose $\{a_n\} \subset [b, c]$ and $\{a_n\} \to a$. Then, $a \subset [b, c]$

4.3.1. Proof

Since $a_n \geq b$, $a_n - b \geq 0$,

$$\{a_n - b\} \to a - b \geq 0$$

Similarly, $c - a \geq 0$, so:

$$b \leq a \leq c$$
$$a \in [b, c]$$

4.4. Closed Sets

A set $A \subset \mathbb{R}$ is closed if whenever a sequence $\{a_n\} \subset A$ converges to a, then $a \in A$.

4.4.1. Example

If A and B are closed, $A \cup B$ is closed.

4.4.2. Example

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$$
$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1)$$

4.5. Open Set

A set $A \in \mathbb{R}$ is open if $\forall x \in A$, $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A$

4.5.1. Example

$\varnothing$ and $\mathbb{R}$ are open and closed.

4.6. Relating Closed and Open Sets

$A \subset \mathbb{R}$ is open iff $\mathbb{R} \setminus A$ is closed.

4.7. Go to ∞

$\forall M > 0$, $\exists N > 0$ such that if $n \geq N$, $a_n > M$

4.7.1. Example

$\{n \cdot (-1)^n\}$ does not go to ∞ and is also not bounded.

4.8. Monotone

A sequence $\{a_n\}$ is monotone if $\forall n \in \mathbb{N}$, $\{a_{n+1}\} \geq a_n$ or $\forall n \in \mathbb{N}, a_{n+1} \leq a_n$. The first is monotone increasing, and the latter is monotone decreasing.

4.9. Monotone Convergence Theorem

Let $\{a_n\}$ be a monotone sequence. $\{a_n\}$ converges iff it is bounded.

If $\{a_n\}$ is monotone increasing then $\{a_n\} \to \sup\{a_n\}$. If $\{a_n\}$ is monotone decreasing, then $\{a_n\} \to \inf\{a_n\}$.

4.9.1. Proof

$(\to)$ Suppose $\{a_n\}$ is monotone and converges. Since $\{a_n\}$ converges, by a theorem it is bounded.

$(\leftarrow)$ Suppose $\{a_n\}$ is monotone increasing and bounded. Because $\{a_n\}$ is bounded, the supremum exists. Let $a = \sup\{a_n\}$.

Fix $\varepsilon > 0$.

Then, there exists N such that $a - \varepsilon < a_N \leq a$.

Since it is monotone increasing, $\forall n \geq N, a_n \geq a_N > a - \varepsilon$. Furthermore, $a_n \leq a$.

Thus, $|a_n - a| = a - a_n \leq a - a_n < \varepsilon$.

Therefore $\{a_n\} \to \sup\{a_n\}$

Similarly if $\{a_n\}$ is monotone decreasing, $\{a_n\} \to \inf\{a_n\}$.

4.9.2. Proposition

The sequence $\left\{\sum_{k=1}^{n} \frac{1}{k}\frac{1}{2^k}\right\}$.

4.9.2.1. Proof

Let $a_n = \sum_{k=1}^{n} \frac{1}{k}\frac{1}{2^k}$. Since $\forall n \in \mathbb{N}, a_{n+1} - a_n = \frac{1}{n+1}\frac{1}{2^{n+1}} > 0$, $a_n + 1 > a_n$

Therefore it is monotone increasing.

$$0 \leq a_n \leq \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} < \frac{1}{1 - \frac{1}{2}} = 2$$

Therefore, $\{a_n\}$ is bounded.

Since it is bounded and monotone increasing, the sequence converges.

4.9.3. Proposition

Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$. $\{a_n\}$ does not converge.

4.9.3.1. Proof

$a_{n+1} - a_n = \frac{1}{n+1} > 0$, so $a_{n+1} > a_n$, and $\{a_n\}$ is monotone increasing.

Claim: $\forall n \in \mathbb{N}, a_{2^n} \geq 1 + \frac{n}{2}$.

Base case ($n = 2$):

$$a_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Suppose $a_{2^n} \geq 1 + \frac{n}{2}$.

Then $a_{2^{n+1}} = a_2$

$$
\begin{aligned}
a_{2^{n+1}} &= a_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \dots + \frac{1}{2^n+2^n} \\
&\geq a_{2^n} + \frac{1}{2^n+2^n} + \frac{1}{2^n+2^n} + \dots + \frac{1}{2^n+2^n} \\
&\geq a_{2^n} + \frac{2^n}{2 \cdot 2^n} \\
&= a_{2^n} + \frac{1}{2} \\
&\geq 1 + \frac{n}{2} + \frac{1}{2} \\
&\geq 1 + \frac{n+1}{2}
\end{aligned}
$$

Thus, $\{a_n\}$ is not bounded and therefore does not converge.

4.10. Nested Interval Theorem

Suppose $A_n = [a_n, b_n]$ for $-\infty < a_n < b_n < \infty$. Suppose that $\forall n \in \mathbb{N}, A_{n+1} \subset A_n$. Then, $\bigcap_{n=1}^{\infty} A_n = [\sup\{a_n\}, \inf\{b_n\}] \neq \varnothing$.

4.10.1. Proof

$\{a_n\}$ is monotone increasing and bounded: $a_1 \leq a_n \leq b_1$. Therefore, it converges to $a = \sup\{a_n\}$. Similarly, $\{b_n\} \to b = \inf\{b_n\}$.

Claim: $\bigcap_{n=1}^{\infty} A_n = [\inf\{b_n\}, \sup\{a_n\}]$ and $a \leq b$.

By a homework problem, $a \leq b$. Let $x \in [a, b]$. $\forall n \in \mathbb{N}, x \geq a \geq a_n \wedge x \leq b \leq b_n$. Therefore, $\forall n \in \mathbb{N}, x \in A_n$, and so $x \in \bigcap_{n=1}^{\infty} A_n$, and so therefore $[a, b] \subset \bigcap_{n=1}^{\infty} A_n$.

Let $x \in \bigcap_{n=1}^{\infty} A_n$.

Therefore, $\forall n \in \mathbb{N}$:

$$
\begin{gathered}
x \in A_n \\
a_n \leq x \leq b_n
\end{gathered}
$$

x is an upper bound of $\{a_n\}$ and a lower bound of $\{b_n\}$, and thisi $x \geq \sup\{a_n\} = a$ and $x \leq \inf\{b_n\} = b$. So $x \in [a, b]$.

4.11. Subsequence

Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing ($n_1 < n_2 < n_3 < \cdots$).

A subsequence of a sequence $\{a_n\}$ is $\{b_k\} = \left\{a_{n_k}\right\}$.

This is usually just denoted by $\left\{a_{n_k}\right\}$.

4.11.1. Example

$\{a_n\} = \left\{\frac{1}{n}\right\}$

Let $n_k = k^2$, and then $\left\{a_{n_k}\right\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\right\}$.

4.12. Subsequences of a Convergent Sequence Converge

If $\{a_n\}$ is a sequence that converges to a, then every subsequence also converges to a.

4.12.1. Proof

Fix $\varepsilon > 0$.

$\exists N_1$ such that $n \geq N_1 \to |a_n - a| < \varepsilon$.

Let $\left\{a_{n_k}\right\}$ be a subsequence of $\{a_n\}$. Then, $\{n_k\}$ is a strictly increasing sequence of natural numbers.

Therefore, $\exists N, k \geq N \to n_k \geq N_1$. Thus, if $k \geq N$, $\left|a_{n_k} - a\right| < \varepsilon$.

4.13. Peak Index

$m \in \mathbb{N}$ is a peak index of a sequence $\{a_n\}$ if $\forall n \geq m, a_m \geq a_n$.

This means that everything in the future is smaller (or the same).

4.14. Every sequence has a monotone subsequence.

Every sequence has a monotone subsequence.

4.14.1. Proof

$\{a_n\}$ has either finitely many peak indices or infinitely many peak indices.

4.14.1.1. Case 1: Finite Peak Indices

This means that there is $N \in \mathbb{N}$ so that there are no peak indices greater than N.

Let $n_1 = N + 1$. For all k, let n_{k+1} be an integer such that $a_{n_{k+1}} > a_{n_k}$. This is possible because there are no peak indices after N, which means that there is always a bigger point in the future.

Then $\left\{a_{n_k}\right\}$ is a monotone increasing subsequence.

4.14.1.2. Case 2: Infinitely Many Peak Indices

Let n_k be the increasing sequence of peak indices. Then, $\left\{a_{n_k}\right\}$ is monotone decreasing.

4.15. Bounded Sequences have convergent subsequences

Every bounded sequence has a convergent subsequence.

4.15.1. Proof

Take a monotone subsequence of the sequence, which is possible by above. Since the sequence is bounded, the subsequence is bounded, and so since the subsequence is monotone and bounded, it is convergent.

4.16. Sequentially Compact

A set S is sequentially compact if every sequence in S has a subsequence that converges in S.

$S \subset \mathbb{R}, \forall\{a_n\} \subset S, \exists\{n_k\}, \left\{a_{n_k}\right\} \to a, a \in S \leftrightarrow S \text{ is sequentially compact.}$

4.16.1. Example

$[0, 1]$

Let $\{a_n\} \subset [0,1]$. Then $\{a_n\}$ is bounded, and therefore there is a subsequence that converges. Call this $\left\{a_{n_k}\right\} \to a$. Since $[0,1]$ is closed, any sequence within converges within, and so $a \in [0,1]$, and therefore $[0,1]$ is sequentially compact.

4.16.2. Non-Examples

- $(0,1)$
  - $\left\{\frac{1}{2n}\right\} \to 0, 0 \notin (0,1)$
- $\mathbb{R}$
  - $\{n\}$ does not converge.

4.17. Bolzano-Weierstrass Theorem

A set S is sequentially compact iff it is closed and bounded.

4.17.1. Proof

$(\to)$ Let S be sequentially compact set.

Let $\{a_n\} \subset S$, so that $\{a_n\} \to a \in \mathbb{R}$. $\exists\left\{a_{n_k}\right\} \to b \in S$. But since $\{a_n\} \to a$, $a = b, b \in S \to a \in S$, and therefore S is closed.

Assume S is not bounded.

$$\forall n \in \mathbb{N}, a_n \in S, |a_n| \geq n$$

Let $\left\{a_{n_k}\right\}$ be a subsequence of $\{a_n\}$.

Let $M \geq 0$. Then if $n \geq M$, $|a_n| \geq n \geq M$.

Then, if $k \geq M$, $n_k \geq k \geq M$, so $\left|a_{n_k}\right| \geq M$. Thus $\left\{a_{n_k}\right\} \to \infty$ and does not converge.

Therefore, S must be bounded.

$(\leftarrow)$ Let S be a closed and bounded set.

Let $\{a_n\} \subset S$. Then $\{a_n\}$ is bounded, and so a subsequence $\left\{a_{n_k}\right\}$ converges to a. Because S is closed $\left\{a_{n_k}\right\} \subset S \to a \in S$.

4.18. Continuous function

A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is continuous if for all $x_0 \in D$ and all $\{x_n\} \subset D$ converging to x_0,

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

4.18.1. Examples

$$f(x) = 3x^2 + 2x - 1$$
$$f : \mathbb{R} \to \mathbb{R}$$

This is a continuous function. Let $x_0 \in \mathbb{R}$. Let $\{x_n\} \to x_0$. Then $\{f(x_n)\} = \left\{3x_n^2 + 2x_n - 1\right\} = 3\{x_n\}^2 + 2\{x_n\} - \{1\} \to x_0$ by various theorems about manipulating sequences.

$$f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}, f : \mathbb{R} \to \mathbb{R}$$

Let $\{x_n\} = \left\{-\frac{1}{n}\right\}$. Then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f\left(-\frac{1}{n}\right) = 1$ but $f(0) = 2$, and therefore $f(x)$ is discontinuous.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

This function is discontinuous everywhere.

4.19. Manipulating Continuous Functions

Suppose $f, g : D \to \mathbb{R}$ are both continuous. Then:

- $f + g$ is continuous
- $f - g$ is continuous
- $f \times g$ is continuous
- $g(x) \neq 0 \to \frac{f}{g}$ is continuous

4.20. Composing Continous Functions

Let $f : U \to D$ and $g : D \to \mathbb{R}$ be continuous. Then $g \circ f$ is also continuous.

4.20.1. Proof

Let $\{x_n\} \subset U$ with $\{x_n\} \to x_0 \in U$. Let $y_n = f(x_n)$. Since f is continuous, $\{f(x_n)\} = \{y_n\} \to f(x_0) := y_0$. Since $f : U \to D$, $\{y_n\} \subset D$ and $y_0 \in D$. Since g is continuous, $\{g(y_n)\} \to g(y_0)$. Thus, $\{g\{f(x_n)\}\} = \{g(y_n)\} \to g(y_0) = g(f(x_0))$, and therefore $g \circ f$ is continuous.

4.21. Maximum and Maximizer

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous. If there $\exists x_o \in D, \forall x \in D, f(x_0) \geq f(x)$, then x_0 is a maximizer of f and $f(x_0)$ is the maximum value.

4.21.1. Examples

- $f(x) = 1, D = \mathbb{R}$
  - maximizers are $\mathbb{R}$
- $f(x) = -x^2, D = \mathbb{R}$
  - maximizers are $\{0\}$, and a max value of 0.
- $f(x) = x, D = (0, 1)$
  - no maximizer
- $f(x) = x, D = \mathbb{R}$
  - no maximizer

4.22. Minimum and Minimizer

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous. If there $\exists x_o \in D, \forall x \in D, f(x_0) \leq f(x)$, then x_0 is a minimizer of f and $f(x_0)$ is the minimum value.

4.23. Image

Let $f : D \subset A \to B$, $f(D) = \{f(x) \mid x \in D\}$.

4.24. Image of a sequentially compact set is sequentially compact

If D is sequentially compact and $f : D \to \mathbb{R}$ is continuous, then $f(D)$ is sequentially compact.

4.24.1. Proof

Let $\{y_n\} \subset f(D)$.

For each n, there exists $x_n \in D$ such that $y_n = f(x_n)$, by definition of image.

Since D is sequentially compact, there exists a subsequence $\left\{x_{n_k}\right\}$ that converges to $x_0 \in D$. This can create a subsequence $\left\{f\left(x_{n_k}\right)\right\} = \left\{y_{n_k}\right\}$.

Since f is continuous, $\lim\limits_{k\to\infty} f\left(x_{n_k}\right) = \lim\limits_{k\to\infty} y_{n_k} = f(x_0) \coloneqq y_0$. Therefore, there exists a subsequence of $\{y_n\}$ that converges in D, and so $f(D)$ is sequentially compact.

4.25. Extreme Value Theorem

If $f : D \to \mathbb{R}$ is continuous, and D is sequentially compact, and f has both a maximizer and a minimizer in D or f attains both its max and minimum values.

4.25.1. Proof

Since D is sequentially compact and f is continuous, $f(D)$ is sequentially compact. We will show that $f(D)$ has a maximum.

Since $f(D)$ is sequentially compact, it is closed and bounded. Therefore, $\sup f(D)$ exists.

$\forall n \in \mathbb{N}$, let $a_n \in f(D)$ such that $\sup f(D) - \frac{1}{n} < a_n \leq \sup f(D)$. Fix $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$. Then if $n \geq N$,

$$0 \leq \sup f(D) - a_n \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$
$$\Rightarrow |\sup f(D) - a_n| < \varepsilon$$

Since $\{a_n\} \to \sup f(D)$ and $f(D)$ is closed, $\sup f(D) \in f(D)$ and thus $f(D)$ contains a maximum value $(\sup f(D))$.

Similarly, $f(D)$ has a minimum value.

4.26. Intermediate Value Theorem

If f is continuous on the interval $[a, b]$, and $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x \in [a, b], f(x) = c$.

4.26.1. Proof

Without loss of generality, $f(a) < f(b)$. Let $f(a) < c < f(b)$.

Let $a_1 = a$, $b_1 = b$. For $n \in \mathbb{N}$, let $m_n = \frac{a_n+b_n}{2}$. If $f(m_n) \leq c$ let $a_{n+1} = m_n, b_{n+1} = b_n$. Otherwise, if $f(m_n) > c$, let $a_{n+1} = a_n, b_{n+1} = m_n$.

We claim that

$$\forall n, a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$$

Base step: $b_1 \geq a_1$.

Inductive step:

If $b_n \geq a_n$, then either $b_{n+1} = b_n$ and $a_{n+1} = \frac{a_n+b_n}{2} < \frac{b_n+b_n}{2} \leq b_{n+1}$ or $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2} \geq \frac{a_n+a_n}{2} = a_{n+1}$.

Outside of induction, if $a_{n+1} = a_n$, then $a_n < a_{n+1}$. If $a_{n+1} = m_n = \frac{a_n+b_n}{2} \geq \frac{a_n+a_n}{2} = a_n$.

This is similarly the case for b_n.

$\{a_n\}$ and $\{b_n\}$ are monotone and bounded, so $\{a_n\} \to a_0$ and $\{b_n\} \to b_0$.

We claim that $\forall n, b_n \geq a_0$.

Suppose $b_n < a_0$. Then, there exists m such that $b_m < a_n \leq a_0$.

Without loss of generality, $m \geq n$. Since the sequences are monotone, $b_m \leq b_n \leq a_m \leq a_0$, and therefore $b_n \geq a_0$. Similarily, $\forall n, a_n \leq b_0$.

Consider $\{b_n - a_n\} \to b_0 - a_0$.

$$b_{n+1} - a_{n+1} = \begin{cases} \frac{a_n+b_n}{2} - a_n \\ b_n - \frac{a_n+b_n}{2} \end{cases} = \frac{b_n - a_n}{2}$$

By induction, $b_n - a_n = \frac{b-a}{2^n}$. $\left\{\frac{b-a}{2^n}\right\} \to 0$, and therefore $0 = b_0 - a_0$. By continuity, $\lim_{n\to\infty} f(a_n) = f(a_0)$ and $\lim_{n\to\infty} f(b_n) = f(b_0)$. By construction, $f(a_n) \leq c$, and $f(b_n) \geq c$. Therefore, $f(a_0) \leq c$, and $f(b_0) \geq c$.

$c \leq f(b_0) = f(a_0) \leq c \to f(b_0) = f(a_0) = c$

4.27. Roots exist

$\forall c > 0, m \in \mathbb{N}, \exists x \in \mathbb{R}, x^m = c$

4.27.1. Proof

$$f(x) = x^m$$

Note $f(0) = 0$ and therefore $0 < c$. Further note $f(c+1) = c^m + ... + mc + 1 > c$ because $m \in \mathbb{N}$ and therefore $m \geq 1$.

By the intermediate value theorem, there exists $x \in [0, c+1]$ such that $x^m = c$.

4.28. Image of an interval is an interval

If I is an interval and f is continuous, then $f(I)$ is an interval.

4.28.1. Proof

Case 1: $I = [a, b]$. Since I is sequentially compact, we can let $\alpha = \min(f(I)), \beta = \max(f(I))$. We claim that $f(I) = [\alpha, \beta]$.

Let $f(x_1) = \alpha, f(x_2) = \beta$. WLOG, assume $x_1 \leq x_2$. Let $\alpha < c < \beta$. Then $\exists x \in [x_1, x_2], [x_1, x_2] \subset [a, b]$ such that $f(x) = c$. Thus, $c \in f(I)$ and $f(I) = [\alpha, \beta]$.

4.29. Uniform Continuity

A function $f : D \to \mathbb{R}$ is uniformly continuous if for all sequences $\{a_n\}, \{b_n\} \subset D$ if $\lim_{n\to\infty} (a_n - b_n) = 0$ then $\lim_{n\to\infty} (f(u_n) - f(v_n))$.

4.29.1. Example

$f(x) = x^2$ is not uniformly continuous.

Let $\{a_n\} = n$ and $\{b_n\} = \left\{n + \frac{1}{n}\right\}$. Then, $\{a_n - b_n\} = \left\{-\frac{1}{n}\right\} \to 0$.

But $\{f(a_n) - f(b_n)\} = \left\{n^2 - \left(n - \frac{1}{n}\right)^2\right\} = \left\{n^2 - n^2 - 2 - \frac{1}{n^2}\right\} = \left\{-2 - \frac{1}{n^2}\right\} \to -2$

Intuitively, the slope increases too fast.

4.29.2. Example

$f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

4.30. Continuous from uniform continuity

Every uniformly continuous function is continuous.

4.30.1. Proof

Let $\{b_n\} = \{x_0\} \subset D$.

Then, $\{a_n\} \to x_0 \leftrightarrow \{a_n - b_n\} \to 0$.

Then, $\{f(a_n) - f(b_n)\} \to 0$, which is equal to $\{f(a_n) - f(x_0)\} \to 0$, so $\{f(a_n)\} \to f(x_0)$, which is equivalent to the definition of continousity.

4.31. Uniform continuity from Continuity

Suppose $f : D \to \mathbb{R}$ is continuous and D is sequentially compact, then f is uniformly continuous.

4.31.1. Proof

Suppose f is not uniformly continuous. Then there exists $\{a_n\}, \{b_n\} \subset D$ such that $\{a_n - b_n\} \to 0$ but $\{f(a_n) - f(b_n)\} \nrightarrow 0$.

Then there exists $\varepsilon > 0$ and subsequences also called $\{a_n\}, \{b_n\}$ such that $|f(a_n) - f(b_n)| > \varepsilon$ for all n.

By sequential compactness, there exists $\left\{a_{n_k}\right\}, \left\{b_{n_k}\right\}$[2] such that $\left\{a_{n_k}\right\} \to a$ and $\left\{b_{n_k}\right\} \to b$, $a, b \in D$. Since $\{a_n - b_n\} \to 0$, $a = b = x_0 \in D$.

Then $\left\{f\left(a_{n_k}\right)\right\} \to \{f(x_0)\}$ and $\left\{f\left(b_{n_k}\right)\right\} \to \{f(x_0)\}$. Then, $\left\{f\left(a_{n_k}\right) - f\left(b_{n_k}\right)\right\} \to f(x_0) - f(x_0) = 0$, but $0 < \varepsilon$, a contradiction.

4.32. Epsilon-Delta Criterion

$f : D \to \mathbb{R}$ satisfies the $\varepsilon - \delta$ criterion at $x_0 \in D$ if $\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta \land x \in D \to |f(x) - f(x_0)| < \varepsilon$.

4.32.1. Example

Show that $f(x) = x^3$ satisfies the $\varepsilon - \delta$ criterion at $x_0 = 2$.

4.32.1.1. Scratch

$$\begin{aligned} & |(x - 2)| \\ = & \left|(x - 2)(x^2 + 2x + 4)\right| \end{aligned}$$

[2]This is questionable, but it does work by first finding a $\left\{a_{n_k}\right\} \to a$, and then finding a subsequence of $\left\{b_{n_k}\right\}$, such that $\left\{b_{n_{k_k}}\right\} \to b$. Since $\left\{a_{n_k}\right\} \to a$, $\left\{a_{n_{k_k}}\right\} \to a$, and so these subsequences do exist

$$= \underbrace{|x^2 + 2x + 4|}_{\text{not huge}} \underbrace{|x - 2|}_{\text{small}}$$

Assume $|x - 2| < 1$. Then,

$$\leq 19|x - 2|$$

4.32.1.2. Proof

Fix $\varepsilon > 0$. Choose $\delta < \min\{1, \frac{\varepsilon}{19}\}$.

Since $|x - x_0| < \delta$, then $|x - x_0| < 1$ and then by above, $|x^3 - 2^3| < 19\delta < 19\frac{\varepsilon}{19} = \varepsilon$

4.33. Relating epsilon-delta criterion and continuity

Let $f : D \to \mathbb{R}$ be continuous iff it satisfies the epsilon-delta criterion at all $x_0 \in D$.

4.34. Uniformly Continuity by $\varepsilon - \delta$ criterion

$f : D \to \mathbb{R}$ is uniformly continuous iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall u, v \in D$, if $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$.

This makes sense because uniformness means that the δ only depends on ε.

4.34.1. Example

Prove that $f(x) = x^2$ is continuous at $x = x_0$ using the $\varepsilon - \delta$ criterion.

Fix $\varepsilon > 0$.

4.34.2. Scratch

If $|x - x_0| < \delta$ Then

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |x - x_0||x + x_0| \\ &< \delta(|x| + |x_0|) \\ &< \delta(2|x_0| + \delta) \\ &< \delta(2|x_0| + 1) \end{aligned}$$

$$\delta(2|x_0| + 1) = \varepsilon \to \delta = \min\left(\frac{\varepsilon}{2|x_0| + 1}, 1\right)$$

4.34.3. Work

Let $\delta = \min\left(\frac{\varepsilon}{2|x_0|+1}, 1\right)$.

$$|x - x_0| < \delta$$

$$|f(x) - f(x_0)| \leq |x - x_0|||x| + f(x_0)| < \delta(2|x_0| + 1) < \varepsilon$$

This would *not* work for uniform continuity since δ depends on x_0.

4.35. Monotone Function

$f : D \to \mathbb{R}$ is monotone increasing if $\forall a, b \in D$ such that $a \leq b$, $f(a) \leq f(b)$ or is monotone decreasing if $\forall a, b \in D$ such that $a \leq b$, $f(a) \geq f(b)$.

4.36. Continuity by image and monotone

If $f : D \to \mathbb{R}$ is monotone such that $f(D)$ is an interval, then f is continuous.

4.36.1. Proof

Suppose $f(D) = I$, but f is not continuous.

Then, there exists a sequence $\{x_n\} \to x_0$ such that $\{f(x_n)\} \nrightarrow f(x_0)$.

Let $y_n = f(x_n)$, $y_0 = f(x_0)$. Then there exists ε and a subsequence y_{n_k} so that $\forall k, \left|y_{n_k} - y_0\right| \geq \varepsilon$.

WLOG, assume f is monotone increasing.

Then, there are subsequences we will also call $\left\{y_{n_k}\right\}$ and $\left\{x_{n_k}\right\}$ such that $\left\{x_{n_k}\right\}$ is monotone. WLOG, assume $\left\{x_{n_k}\right\}$ is monotone increasing.

Since $f\left(x_{n_k}\right) = y_{n_k} < y_0 - \varepsilon$ and $f(x_0) = y_0$, there must exist a $x^* \in (x_n, x_0)$ such that $f(x^*) = y_0 - \varepsilon$.

Since $\left\{x_{n_k}\right\} \to x_0$, there is a k such that $x^* < x_{n_k} < x_0$ but $f\left(x_{n_k}\right) = y_{n_k}$ and $y_{n_k} < y_0 - \varepsilon$, so this shouldn't be true and therefore there is a contradiction.

4.37. Strictly Monotone Function

A function $f : D \to \mathbb{R}$ is strictly monotone increasing if whenever $x, y \in D$ such that $x < y$, then $f(x) < f(y)$. For strictly decreasing, $x < y$ implies $f(x) > f(y)$.

Put simply, monotone but cannot go sideways.

4.38. Strictly monotone functions are injective

A strictly monotone function is injective.

4.38.1. Proof

Let $f : D \to \mathbb{R}$ be strictly monotone. WLOG, assume it is increasing.

Let $x, y \in D$ such that $f(x) = f(y)$. On the contrary, assume $x \neq y$. WLOG, assume $x < y$.

Since $x < y$, $f(x) < f(y)$, but $f(x) = f(y)$ so this is impossible and therefore a contradiction.

Therefore $x = y$, and f is injective.

4.39. Strictly monotone functions can be bijective

If f is strictly monotone, then $f : D \to f(D)$ is bijective.

4.40. Strictly monotone functions can have an inverse

A strictly monotone function $f : D \to f(D)$ has an inverse f^{-1}.

4.41. Inverses are continuous

If D is an interval and $f : D \to f(D)$ is strictly monotone, then $f^{-1} : f(D) \to D$ is continuous.

4.41.1. Proof

Firstly, f^{-1} exists by above, and $f^{-1}(f(D)) = D$.

Let $a, b \in f(D)$ such that $a \leq b$. WLOG, assume f is monotone increasing.

Let $x = f^{-1}(a)$ and $y = f^{-1}(b)$. Therefore, $a = f(x)$ and $b = f(y)$. Therefore, $a \leq b$ and so $f(x) \leq f(y)$ and therefore $x \leq y$.

Thus $f^{-1}(a) < f^{-1}(b)$. Since f^{-1} is monotone and $f^{-1}(f(D))$ is an inverval, f^{-1} is continuous.

4.42. $x^{\frac{m}{n}}$ exists and define x^r

Let $n \in \mathbb{N}$ and $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^n$. Then, f is strictly monotone increasing.

f^{-1} exists and is denoted $f^{-1}(x) = x^{\frac{1}{n}}$. It is continuous. Then, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$.

Let $r \in (0, \infty)$.

Let $\{a_n\} \subset \mathbb{Q}$ such that $\{a_n\} \to r$. Define $x^r = \lim_{n \to \infty} x^{a_n}$.[3]

4.43. Limit Point

Let $D \subset \mathbb{R}$. $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\}$ such that $\{x_n\} \to x_0$.

4.44. Limit

$f : D \to \mathbb{R}$ has a limit L as x approaches a limit point x_0 if for all sequences $\{x_n\} \subset D \setminus \{x_0\}$ such that $\{x_n\} \to x_0$, $\{f(x_n)\} \to L$.

This is written:

$$\lim_{x \to x_0} f(x) = L$$

4.45. Algebraic Limit Theorem

If $\lim_{x \to x_0} f(x) = L_1$, $\lim_{x \to x_0} f(x) = L_2$:

$$\lim_{x \to x_0} f(x) + g(x) = L_1 + L_2$$
$$\lim_{x \to x_0} f(x) - g(x) = L_1 - L_2$$
$$\lim_{x \to x_0} f(x)g(x) = L_1 L_2$$
$$L_2 \neq 0 \to \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$

4.46. Continuous Functions and Limits

Iff $f : D \to \mathbb{R}$ is continuous and $x_0 \in D$ and x_0 is a limit point of D

$$\lim_{x \to x_0} f(x) = f(x_0)$$

[3] Unproven that this exists but oh well it does.

5. Derivatives

5.1. Limit Definition of the Derivative

$f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$ if x_0 is a limit point of D and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and $f'(x_0)$ is the derivative.

5.1.1. Example

Find $f'(x)$ for $f(x) = mx + b$.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{mx + b - mx_0 + b}{x - x_0} = \lim_{x \to x_0} m \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} m = m$$

Therefore $f'(x) = m$.

5.1.2. Example

Find $f'(x)$ for $f(x) = x^n$.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)\left(x^{n-1} + x_0 x^{n-2} + x_0^2 x^{n-3} + \cdots + x_0^{n-2} x + x_0^{n-1}\right)}{x - x_0} \\ &= \lim_{x \to x_0} \left(x^{n-1} + x_0 x^{n-2} + x_0^2 x^{n-3} + \cdots + x_0^{n-2} x + x_0^{n-1}\right) \\ &= x_0^{n-1} + x_0 x_0^{n-2} + x_0^2 x_0^{n-3} + \cdots + x_0^{n-2} x_0 + x_0^{n-1} \\ &= n x_0^{n-1} \end{aligned}$$

5.2. Differentiability implies continuity

A differentiable function is continuous.

5.2.1. Proof

Let $f : D \to \mathbb{R}$ be differentiable.

Then, $\forall x_0 \in D$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}$$

Furthermore, $\lim_{x \to x_0} x - x_0 = 0$

Then:

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\ &= \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right) 0 = 0 \end{aligned}$$

Since $f(x_0) - f(x_0) = 0$ at each point, $f(x)$ is continuous.

5.2.2. Example: It's not the other way

Show $|x|$ is not differentiable at $x_0 = 0$.

$$\lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}$$

Take the sequences $\{a_n\} = \{1/n\}$ and $\{b_n\} = \{-1/n\}$.

$$\left\{\frac{|a_n|}{a_n}\right\} = \left\{\frac{\left|\frac{1}{n}\right|}{\frac{1}{n}}\right\} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$
$$\left\{\frac{|b_n|}{b_n}\right\} = \left\{\frac{\left|-\frac{1}{n}\right|}{-\frac{1}{n}}\right\} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1$$

But $1 \neq -1$, so the limit does not exist.

5.3. Combining Derivatives

If $f, g : D \to \mathbb{R}$ that are differentiable, then:

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}$, if $g(x) \neq 0$
4. $\left(\frac{f}{g}\right)'(x) = \frac{f'g(x) - f(x)g'(x)}{(g(x))^2}$, if $g(x) \neq 0$

5.3.1. Proof

Let $x_0 \in D$.

$$\begin{aligned}(f+g)'(x_0) &= \lim_{x\to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) + g'(x_0)\end{aligned}$$

$$\begin{aligned}(fg)'(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x)g(x) + f(x)g(x_0) - f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0)}{x - x_0} + \frac{f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to x_0} f(x)\frac{g(x)-g(x_0)}{x-x_0} + g(x_0)\frac{f(x)-f(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} f(x) \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} + \lim_{x\to x_0} g(x_0) \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\
&= f(x_0)g'(x) + g(x_0)f'(x_0)
\end{aligned}
$$

5.4. Neighborhood

Let $x_0 \in \mathbb{R}$. $I \subset \mathbb{R}$ is a neighborhood of x_0 if I is an open interval x_0.

5.5. Change of variables for limits

Let $x_0 \in \mathbb{R}$, I be a neighborhood of x_0, and $f : I \to \mathbb{R}$ be continuous.

Then, if $y_0 = f(x_0)$ and $\lim\limits_{y\to y_0} g(y)$ exists,

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to y_0} g(y)$$

5.6. Proof

Let $J = f(I)$. J is an interval that contains y_0.

Let $\{x_n\} \subset I$, $\{x_n\} \to x_0$.

Let $y_n = f(x_n)$. $\{y_n\} \to y_0$ due to continuity.

We know $\lim\limits_{y\to y_0} g(y) = L$, so $\{g(y_n)\} \to L$.

Furthermore, $\{g(f(x_n))\} \to L$

So $\lim\limits_{x\to x_0} g(f(x)) = L$.

5.7. Invertibility for change of variables

If $f : I \to \mathbb{R}$ is continuous and invertible then, $\forall x_0 \in I, y_0 = f(x_0)$, then

$$\lim_{x\to x_0} g(f(x)) = \lim_{x\to x_0} g(y)$$

5.8. Derivative of Inverse

Suppose $x_0 \in \mathbb{R}$, I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable with $f'(x_0) \neq 0$, then if f is invertible:

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

5.8.1. Proof

Let $y = f(x)$.

$$
\begin{aligned}
\left(f^{-1}\right)'(y_0) &= \lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\
&= \lim_{y\to y_0} \frac{x - x_0}{f(x) - f(x_0)}
\end{aligned}
$$

$$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)}$$
$$= \frac{1}{\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$
$$= \frac{1}{f'(x_0)}$$

5.9. Derivative of Root

$$f(x) = x^n$$
$$f^{-1}(x) = x^{1/n}$$
$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}$$
$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$
$$\left(f^{-1}\right)'(y) = \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}$$

5.10. Chain rule

Suppose $f : I \to \mathbb{R}$ and $g : f(I) \to \mathbb{R}$ are differentiable.

Then,

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

5.10.1. Proof

Let $x_0 \in I$, $y = f(x)$, $y_0 = f(x_0)$.

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

If a neighborhood I' of x_0, $I' \subset I$ where f is invertible exists, then

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$
$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}$$
$$= f'(x_0)g'(y_0) = f'(x_0)g'(f(x_0))$$

If it is not invertible at $x = x_0$, then there $\{y_n\} \to x_0$ and $\{z_n\} \to x_0$, $\{y_n\}, \{z_n\} \subset I$ such that $y_n \neq z_n$ but $f(y_n) = f(z_n)$ for all $n \in \mathbb{N}$.

Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{n \to \infty} \frac{f(y_n) - f(x_0)}{y_n - x_0}$$

$$= \lim_{n \to \infty} \frac{f(z_n) - f(x_0)}{z_n - x_0}$$

5.11. Power Rule (for rationals)

Let $r \in \mathbb{Q}$, $r \geq 0$. Let $f(x) = x^r$. Then, $f'(x) = rx^{r-1}$.

5.11.1. Proof

Let $r = \frac{m}{n}$. Let $g(x) = x^m$ and $h(x) = x^{\frac{1}{n}}$. Then, $f(x) = g(h(x))$.

By the chain rule, $f'(x) = h'(x)g'(h(x))$.

$g'(x) = mx^{m-1}$, $h'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$

$$\begin{aligned} f'(x) &= \frac{1}{n}x^{\frac{1}{n}-1} \cdot m\left(x^{\frac{1}{n}}\right)^{m-1} \\ &= \frac{m}{n}x^{\frac{1}{n}-1}x^{\frac{m}{n}-\frac{1}{n}} \\ &= \frac{m}{n}x^{\frac{m}{n}-1} \\ &= rx^{r-1} \end{aligned}$$

5.12. The Derivative is Zero at a Maximzer

Suppose $f : [a, b] \to \mathbb{R}$ is differentiable with maximizer $x_0 \in (a, b)$. Then, $f'(x_0) = 0$.

5.12.1. Proof

$$\begin{aligned} f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} \end{aligned}$$

If $x < x_0$, $f(x) - f(x_0) \leq 0$, and $x - x_0 < 0$, and so therefore:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Therefore:

$$\lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Alternatively, if $x > x_0$. $f(x) - f(x_0) \geq 0$, $x - x_0 > 0$, and so therefore:

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

Therefore $f'(x_0) = 0$.

5.13. Rolle's Theorem

Suppose $f : I \to \mathbb{R}$ is differentiable, and suppose $a, b \in I$ such that $a < b$, and $f(a) = f(b)$. Then, $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

5.13.1. Proof

f is continuous on $[a, b]$. Let $m = \min_{x \in [a,b]} f(x)$, $M = \max_{x \in [a,b]} f(x)$.

Case 1: $m = M$

Then $f(x)$ is constant for all $x \in (a, b)$, so $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: $f(a) = m \vee f(b) = m, m \neq M$

Then, $M = f(x_0)$, $x_0 \in (a, b)$.

Case 3: $f(a) = M \vee f(b) = M, m \neq M$.

Then, the maximum of g, $g = -f$, occurs at $x_0 \in (a, b)$, and $f'(x_0) = -g'(x_0)$.

Case 4: $M \in f(x_0), x_0 \in (a, b)$:

By the lemma, $f'(x_0) = 0$, or in case 3, $g'(x_0) = 0 \Rightarrow f'(x_0) = 0$.

5.14. Mean Value Theorem

Let $f : I \to \mathbb{R}$ be differentiable. Let $a, b \in I$, $a < b$.

Then, $\exists x_0 \in (a, b)$, such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$

5.14.1. Proof

Let $g(x) = f(x) - x\frac{f(b)-f(a)}{b-a}$

$g : I \to \mathbb{R}$ is differentiable and $g(a) = g(b)$.

$$g(a) = f(a) - a\frac{f(b) - f(a)}{b - a} = \frac{f(a)(b - a) + af(b) - af(a)}{b - a} = \frac{bf(a) - af(a) + af(b) - af(a)}{b - a}$$

$$g(b) = f(b) - b\frac{f(b) - f(a)}{b - a} = \frac{f(b)(b - a) + bf(b) - bf(a)}{b - a} = \frac{bf(b) - af(b) + bf(b) - bf(a)}{b - a}$$

By Rolle's theorem, $g'(x_0) = 0$. $f'(x_0) = g'(x_0) + \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$.

5.15. Identity Criterion

$f : I \to \mathbb{R}$ is constant iff $f' \equiv 0$.

5.15.1. Proof

$(\to)$ If $f(x) = \mathbb{c}$, then

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\mathbb{c} - \mathbb{c}}{x - x_0} = \lim_{x \to x_0} 0 = 0$$

$(\leftarrow)$ Suppose $f'(x) = 0$. Choose $x_0 \in I$ and let $f(x_0) = \mathbb{c}$.

Let $x \in I$. Then, by the MVT, $\exists x_1 \in (x, x_0)$ such that $f'(x_1) = \frac{f(x_0)-f(x)}{x_0-x} = 0$.

Thus, $f(x) = f(x_0) = \mathbb{c}$ for all x.

5.16. Equal derivatives differ by a constant.

If $f, g : I \to \mathbb{R}$ are differentiable and $f'(x) = g'(x)$ for all $x \in I$, then $\exists c \in \mathbb{R}$ such that $f(x) = g(x) + c$

5.16.1. Proof

Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$. Then by the lemma, $h(x) = c$ for some $c \in \mathbb{R}$. Thus, $f(x) = g(x) + c$.

5.17. Strictly Increasing by derivative

Suppose $f : I \to \mathbb{R}$ is differentiable, then $f(x)$ is strictly monotone increasing if $f'(x) > 0$ for all $x \in I$.

5.17.1. Proof

Suppose $f'(x) > 0$ for all x. Then, let $u, v \in I$, $u < v$. Then, by the MVT, $\exists x_0 \in (u, v)$ such that

$$f'(x_0) = \frac{f(v) - f(u)}{v - u} > 0$$

Thus $f(v) > f(u)$.

5.17.2. Example

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x == 0 \end{cases}$$

- $f'(0) = 0$
- f is not monotone near 0
- $f'(x)$ is not continuous.

5.17.3. Example

Show that $1 + x + x^5 = 0$ has exactly one solution.

$f(x) = 1 + x + x^5$ is continuous on the real numbers.

$$f(0) = 1, f(-1) = -1$$

By the IVT, $\exists x_0 \in [-1, 0], f(x_0) = 0$

Suppose there exists $x_1 \neq x_0$ such that $f(x_1) = 0$. Since f is differentiable, we can apply the MVT to see that there exists $z \in (x_0, x_1)$.

$$f'(z) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 0$$

But $f'(x) = x^4 + 1 > 0$, so $f'(z) > 0$, but $f'(z) = 0$, a contradiction, and therefore there is at most one root.

5.18. Cauchy Mean Value Theorem

Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b) and $\forall x \in (a, b), g'(x) \neq 0$. Then, $\exists x_0 \in (a, b)$:

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5.18.1. Proof

Let

$$h(x) = f(x) - g(x)\frac{f(b) - f(a)}{g(b) - g(a)}$$

$$g(b) - g(a) \neq 0$$

because otherwise, there exists $x_1 \in (a, b)$ such that $g'(x_1) = 0$ by Rolle's theorem.

Note that $h(a) = h(b)$, which can be verified with algebra.

By Rolle's theorem, $\exists x_0 \in (a, b)$ such that $h'(x_0) = 0$, and therefore $f'(x_0) - g'(x_0)\frac{f(b)-f(a)}{g(b)-g(a)} = 0$

And finally:

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5.19. Vague Taylor-series-like statement

Let $f : I \to \mathbb{R}$ be n-times differentiable such that for some point $x_0 \in I$

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Then for any $x \neq x_0$, $x \in I$, there exists z between x an x_0, such that

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

5.19.1. Proof

Let $g(x) = (x - x_0)^n$. By the CMVT, there exists $x_1 \in (x_0, x)$ such that $\frac{f'(x_1)}{g'(x_1)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)}{g(x)}$

By the CMVT applied to f' and g' in (x_0, x_1). Then, there exists $x_2 \in (x_0, x_1)$:

$$\frac{f''(x_2)}{g''(x_2)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} = \frac{f'(x_1)}{g'(x_1)} = \frac{f(x)}{g(x)}$$

Apply n times to get $x_n \in (x_0, x)$ such that

$$\frac{f^{(n)}(x_n)}{g^{(n)}(x_n)} = \frac{f(x)}{g(x)}$$

$$\frac{f^{(n)}(x_n)}{n!} = \frac{f(x)}{(x - x_0)^n}$$

$$f(x) = \frac{f^{(n)}(x_n)}{n!}(x - x_0)^n$$

$z = x_n$, and we have proven the theorem.

6. Integrals

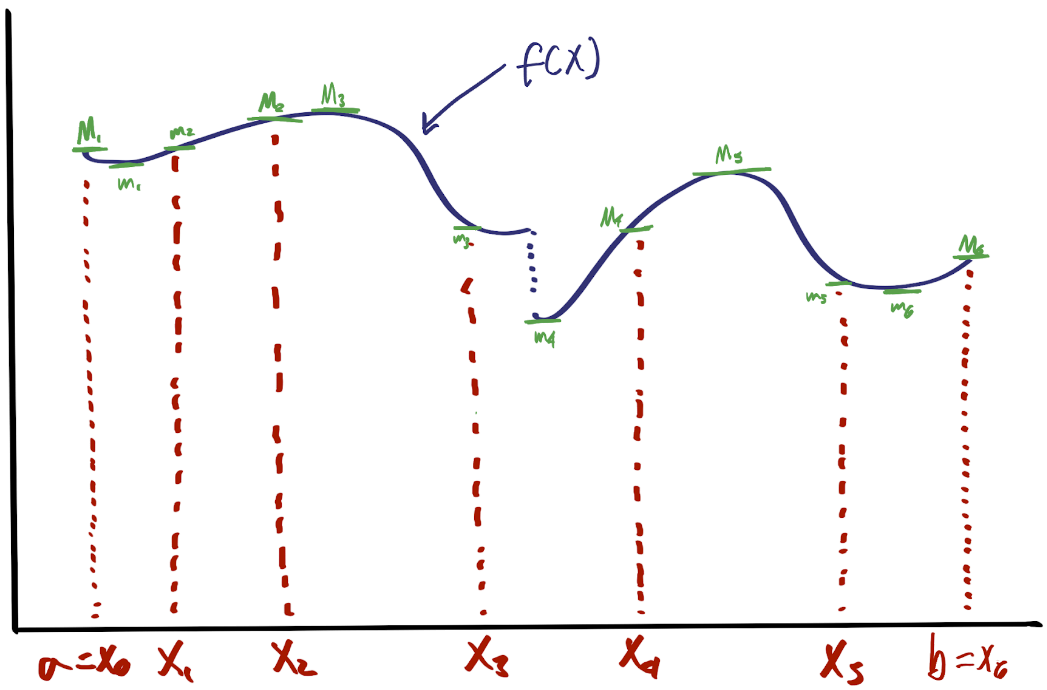
6.1. Darboux Sums

For some function $f : [a, b] \to \mathbb{R}$, let P be a partition of P, such that

$$P = [x_0, x_1, \cdots, x_n]$$
$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

Then, define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$
$$m_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$



Assume f is bounded. Then, let

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$
$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

U is called the upper sum and L is called the lower sum.

6.2. Upper and Lower Integrals

Let $f : [a, b] \to \mathbb{R}$ be bounded. Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. Then

$$\overline{\int_a^b} f = \inf_{P \in \mathcal{P}} U(f, P)$$

$$\underline{\int_a^b} f = \sup_{P \in \mathcal{P}} L(f, P)$$

$\overline{\int}$ denotes the upper integral and $\underline{\int}$ denotes the lower integral.

Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

6.2.1. Proof

Let $P_1, P_2 \in \mathcal{P}$. Let P^* be a refinement of P_1 and P_2. Then

$$U(f, P_1) \geq U(f, P^*) \geq L(f, P^*) \geq L(f, P_2)$$

6.2.2. Counterexample to equality

The Dirichlet function is a counterexample and is defined as

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Then, $\overline{\int_a^b} f = b - a$ and $\underline{\int_a^b} f = 0$ since $\inf_{x \in [x_{i-1}, x_i]} f(x) = 0$ and $\sup_{x \in [x_{i-1}, x_i]} f(x) = 1$ due the density of the rationals in the reals.

But $b - a \neq 0$ for $b \neq a$, which is very possible.

6.3. Integral

$f : [a, b] \to \mathbb{R}$ is integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f$. Then,

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$$

6.4. Lemma

For any partition P and a integrable function $f : [a, b] \to \mathbb{R}$, $L(f, P) \leq \int_a^b f \leq U(f, P)$

6.5. Archimedes-Riemann

$f : [a, b] \to \mathbb{R}$ is integrable iff there exists a sequence of partitions P_n such that

$$\lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = 0$$

Moreover, for such a sequence,

$$\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n) = \int_a^b f$$

6.5.1. Proof

$(\leftarrow)$ If P_n exists such that

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$$

then,

$$\inf_{p\in\mathcal{P}} U(f,p) \le \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) \le \sup_{p\in\mathcal{P}} L(f,P) \le \inf_{p\in\mathcal{P}} U(f,p)$$

Therefore:

$$\inf_{p\in\mathcal{P}} U(f,p) = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = \sup_{p\in\mathcal{P}} L(f,P) = \inf_{p\in\mathcal{P}} U(f,p)$$

And therefore

$$\sup_{p\in\mathcal{P}} L(f,P) = \inf_{p\in\mathcal{P}} U(f,P) = \int_a^b f$$

And so f is integrable.

$(\rightarrow)$ Let f be integrable. Then,

$$\int_a^b f = \inf_{p\in\mathcal{P}} U(f,P)$$

Let $n \in \mathbb{N}$. Therefore, there exists a partition $P' \in \mathcal{P}$ such that $U(f,P') < \int_a^b f + \frac{1}{n}$

Similarly, there exists $P'' \in \mathcal{P}$ such that

$$L(f,P'') > \int_a^b f - \frac{1}{n}$$

Let P_n be a refinement of P' and P''.

Therefore,

$$U(f,P_n) - L(f,P_n) < \int_a^b f + \frac{1}{n} - \int_a^b f + \frac{1}{n} = \frac{2}{n}$$

Therefore, we know:

$$0 \le U(f,P_n) - L(f,P_n) < \frac{2}{n}$$

Since $L(f,P_n) \le U(f,P_n)$.

Therefore

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$$

6.6. Monotone Functions are Integrable

Monotone functions are integrable.

6.6.1. Proof

Let $f : [a, b] \to \mathbb{R}$ be monotone. Let P_n be a partition with $n + 1$ equally spaced points. WLOG, assume f is monotone increasing.

Then

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i) \left(\frac{b-a}{n}\right) \end{aligned}$$

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_{i-1}) \left(\frac{b-a}{n}\right) \end{aligned}$$

Then

$$\begin{aligned} &U(f, P_n) - L(f, P_n) \\ &= \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) - \sum_{i=1}^{n} f(x_{i-1})\left(\frac{b-a}{n}\right) \\ &= \left(\frac{b-a}{n}\right)\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(x_{i-1})\right) \\ &= \left(\frac{b-a}{n}\right)\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=0}^{n-1} f(x_i)\right) \\ &= \left(\frac{b-a}{n}\right)(f(x_n) - f(x_0)) \\ &= \left(\frac{b-a}{n}\right)(f(b) - f(a)) \end{aligned}$$

And

$$\lim_{n \to \infty} \frac{(b-a)(f(b) - f(a))}{n} = 0$$

Therefore

$$\int_a^b f = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

6.7. Additivity of the Integral

If $f : [a, b] \to \mathbb{R}$ is integrable and $c \in (a, b)$ then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

6.7.1. Proof

Since f is integrable, there exists a sequence of partitions $\{P_n\}$ such that

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$$

and moreover

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n)$$

WLOG, by the refinement lemma, assume that $\forall m, c \in P_n$.

Let $P_n' = P_n \cap [a, c]$ and $P_n'' = P_n \cap [c, b]$.

By the definition of Darboux Sums,

$$U(f, P_n) = U(f, P_n') + U(f, P_n'')$$
$$L(f, P_n) = L(f, P_n') + L(f, P_n'')$$

$$\begin{aligned} &\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) \\ &= \lim_{n\to\infty} ((U(f, P_n') + U(f, P_n'')) - (L(f, P_n') + L(f, P_n''))) \\ &= \lim_{n\to\infty} ((U(f, P_n') - L(f, P_n')) + (U(f, P_n'') - L(f, P_n''))) \end{aligned}$$

Since $(U(f, P_n') - L(f, P_n')) > 0$ and $(U(f, P_n'') - L(f, P_n'')) > 0$, the two separate limits go to zero if the whole limit goes to zero.

$$0 = \lim_{n\to\infty} (U(f, P_n') - L(f, P_n'))$$
$$0 = \lim_{n\to\infty} (U(f, P_n'') - L(f, P_n''))$$

Then, by the Archimedes-Riemann theorem, $\lim_{n\to\infty} U(f, P_n') = \int_a^c f$ and $\lim_{n\to\infty} U(f, P_n'') = \int_c^b f$, and those limits exist.

Therefore

$$\begin{aligned} \int_a^b f &= \lim_{n\to\infty} U(f, P_n) \\ &= \lim_{n\to\infty} U(f, P_n') + U(f, P_n'') \\ &= \lim_{n\to\infty} U(f, P_n') + \lim_{n\to\infty} U(f, P_n'') \\ &= \int_a^c f + \int_c^b f \end{aligned}$$

6.8. Monotonicity of the Integral

If $f, g : [a, b] \to \mathbb{R}$ with $\forall x, f(x) \leq g(x)$ and both integrable, then

$$\int_a^b f \leq \int_a^b g$$

6.8.1. Proof

There exists a sequence $\{P_n\}$ for both f and g such that

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n)$$
$$\int_a^b g = \lim_{n\to\infty} U(g, P_n)$$

Since $f \leq g$, $U(f, P_n) \leq U(g, P_n)$ for all n.

Therefore

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) \leq \lim_{n\to\infty} U(g, P_n) = \int_a^b g$$

6.9. Linearity of the Integral

Let $f, g : [a, b] \to \mathbb{R}$ be integrable and $\alpha, \beta \in \mathbb{R}$. Then,

$$\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b f$$

6.9.1. Lemma

Let $f, g : [a, b] \to \mathbb{R}$ be integrable (bounded?), $\alpha \in \mathbb{R}$, and P be a partition of $[a, b]$. Then,

$$L(f, P) + L(g, P) \leq L(f + g, P)$$
$$U(f + g, P) \leq U(f, P) + U(g, P)$$

If $\alpha \geq 0$

$$U(\alpha f, P) = \alpha U(f, P)$$
$$L(\alpha f, P) = \alpha L(f, P)$$

If $\alpha < 0$

$$U(\alpha f, P) = \alpha L(f, P)$$
$$L(\alpha f, P) = \alpha U(f, P)$$

6.9.1.1. Proof

For any bounded function h, define

$$M_i(h) = \sup_{x\in[x_{i-1}, x_i]} h$$

$$m_i(h) = \inf_{x \in [x_{i-1}, x_i]} h$$

Observe

$$\forall x \in [x_{i-1}, x_i], f(x) + g(x) \leq M_i(f) + M_i(g)$$

Therefore

$$M_i(f+g) \leq M_i(f) + M_i(g)$$

Therefore

$$U(f+g, P) \leq U(f, P) + U(g, P)$$

If $\alpha \geq 0$, then $M_i(\alpha f) = \sup_{x \in [x_{i-1}, x_i]} \alpha f(x) = \alpha \sup_{x \in [x_{i-1}, x_i]} f(x) = \alpha M_i(f)$.

Similarly $m_i(\alpha f) = \alpha m_i(f)$.

If $\alpha < 0$ then $M_i(\alpha f) = \sup_{x \in [x_{i-1}, x_i]} \alpha f(x) = \alpha \inf_{x \in [x_{i-1}, x_i]} f(x) = \alpha m_i(f)$.

Similarly $m_i(\alpha f) = \alpha M_i(f)$.

6.9.2. Proof

Case 1: $\beta = 0$

There exists a sequence $\{P_n\}$ of partitions such that $\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0$.

Then

$$\begin{aligned} &\lim_{n \to \infty} U(\alpha f, P_n) - L(\alpha f, P_n) \\ &= |\alpha| \lim_{n \to \infty} U(f, P_n) - L(f, P_n) \\ &= 0 \end{aligned}$$

$$\int_a^b \alpha f = \lim_{n \to \infty} U(\alpha f, P_n) = \begin{cases} \alpha \lim_{n \to \infty} U(f, P_n) & \alpha \geq 0 \\ \alpha \lim_{n \to \infty} L(f, P_n) & \alpha < 0 \end{cases} = \alpha \int_a^b f$$

Case 2: $\alpha = \beta = 1$

There exists a sequence $\{P_n\}$ for f, g.

$$L(f, P_n) + L(g, P_n) \leq L(f+g, P_n) \leq U(f+g, P_n) \leq U(f, P_n) + U(g, P_n)$$

Then as $n \to \infty$,

$$L(f, P_n) + L(g, P_n) = \int_a^b f + \int_a^b g$$

$$U(f, P_n) + U(g, P_n) = \int_a^b f + \int_a^b g$$

Therefore

$$\int_a^b (f+g) = L(f+g, P_n) = U(f+g, P_n) = \int_a^b f + \int_a^b g$$

Case 3: Full thing

$$\int_a^b (\alpha f + \beta g) = \int_a^b \alpha f + \int_a^b \beta g = \alpha \int_a^b f + \beta \int_a^b g$$

6.10. Gap

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition. Then, $\text{gap } P = \max_{i=1,...,n} (x_i - x_{i-1})$

6.11. All continuous functions are integrable

All continuous functions on $[a, b]$ are integrable.

6.11.1. Lemma

Let $f : [a, b] \to \mathbb{R}$ be continuous and let P be a partition of $[a, b]$. Then, there exists u, v in one partition subinterval such that

$$0 \leq U(f, P) - L(f, P) \leq (f(u) - f(b))(b - a)$$

6.11.1.1. Proof

$P = \{x_0, x_1, ..., x_n\}$

Since f is continuous on the interval $[x_{i-1}, x_i]$, and $[x_{i-1}, x_i]$ is sequentially compact, by the extreme value theorem, there exists u_i, v_i such that $f(u_i) = \sup_{x \in [x_{i-1}, x_i]} f(x) = M_i$, and $f(v_i) = \inf_{x \in [x_{i-1}, x_i]} f(x) = m_i$.

For some i_0, $f\left(u_{i_0}\right) - f\left(v_{i_0}\right) = M_{i_0} - m_{i_0} = \max_{i=1,...,n} M_i - m_i$.

Then

$$\begin{aligned} & U(f, P) - L(f, P) \\ & \qquad = \sum_{i=1}^n M_i (x_i - x_{i-1}) - \sum_{i=1}^n m_i (x_i - x_{i-1}) \\ & \qquad = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ & \qquad \leq \sum_{i=1}^n \left(M_{i_0} - m_{i_0}\right)(x_i - x_{i-1}) \\ & \qquad = \left(f\left(u_{i_0}\right) - f\left(v_{i_0}\right)\right) \sum_{i=1}^n (x_i - x_{i-1}) \\ & \qquad = \left(f\left(u_{i_0}\right) - f\left(v_{i_0}\right)\right)(b - a) \end{aligned}$$

6.11.2. Proof

Let $f : [a, b] \to \mathbb{R}$ be continuous.

Let $\{P_n\}$ be any sequence of partitions such that $\lim_{n \to \infty} \text{gap } P_n = 0$.

Then, there exists $\{u_n\}, \{v_n\}$ such that

$$0 \leq U(f, P_n) - L(f, P_n) \leq (f(u_n) - f(v_n))(b - a)$$

Note that $|u_n - v_n| \leq \text{gap } P_n$, therefore $\{u_n - v_n\} \to 0$

Because f is continuous on a sequentially compact domain, it is uniformly continuous, so $\{f(u_n) - f(v_n)\} \to 0$.

Therefore, by the comparison lemma, $\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0$

6.12. Boundary does not matter

Let $f : [a, b] \to \mathbb{R}$ be bounded and be continuous on (a, b). Then, f is integrable and $\int_a^b f$ does not depend on $f(a)$ or $f(b)$.

6.12.1. Proof

Let $\{a_n\} \to a$ and $\{b_n\} \to b$ with $a < a_n < b_n < b$.

f is continuous on $[a_n, b_n]$, so it is integrable on $[a_n, b_n]$.

Since it is integrable, there exists a sequence of partitions $\{P_n\}$ of $[a_n, b_n]$ such that $0 \leq U(f, P_n) - L(f, P) < \frac{1}{n}$[4]

Since f is bounded, there exists B such that $\forall x \in [a, b], -B \leq f(x) \leq B$.

Let P_n^* be a partition of $[a, b]$ be formed by adding a and b to P_n.

In other words, if $P_n = \{a_n = x_0, x_1, ..., x_{k-1}, x_k = b_n\}$, then $P_n^* = \{a, a_n, x_1, ..., x_{k-1}, b_n, b\}$.

Then

$$C_n = \left(\sup_{x \in [a, a_n]} f(x) - \inf_{x \in [a, a_n]} f(x) \right) (a_n - a)$$

$$D_n = \left(\sup_{x \in [b_n, b]} f(x) - \inf_{x \in [b_n, b]} f(x) \right) (b - b_n)$$

$$\begin{aligned} |C_n| &\leq 2B(a_n - a) \\ |D_n| &\leq 2B(b_n - b) \end{aligned}$$

$$\begin{aligned} 0 &\leq U(f, P_n^*) - L(f, P_n^*) \\ &= U(f, P_n) - L(f, P_n) + C_n + D_n \\ &\leq \frac{1}{n} + 2B((a_n - a) + (b - b_n)) \end{aligned}$$

Since $\left\{\frac{1}{n} + 2B((a_n - a) + (b - b_n))\right\} \to 0$, $\lim_{n \to \infty} U(f, P_n^*) - L(f, P_n^*) = 0$ by the comparision lemma.

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[4] For each n, there exists a sequence of partitions $\{P_n^m\}$ such that $\lim_{m \to \infty} U(f, P_n^m) - L(f, P_n^m) = 0$. Then, by the definition of the limit, there exists $M > 0$ such that $0 \leq U(f, P_n^m) - L(f, P_n^m) < \frac{1}{n}$. Let $P_n = P_n^M$.

Therefore f is integrable.

Furthermore

$$\begin{aligned}
&\int_a^b f \\
&= \lim_{n\to\infty} U(f, P_n^*) \\
&= \lim_{n\to\infty} U(f, P_n) + \left(\sup_{x\in[a,a_n]} f(x)\right)(a_n - a) + \left(\sup_{x\in[b_n,b]} f(x)\right)(b - b_n) \\
&= \lim_{n\to\infty} U(f, P_n)
\end{aligned}$$

Therefore, f does not dependent on $f(a)$ or $f(b)$.

6.13. First Fundamental Theorem of Calculus

Let $F : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) with $F' : (a, b) \to \mathbb{R}$ continuous and bounded. Then, F' is integrable and

$$\int_a^b F' = F(b) - F(a)$$

6.13.1. Lemma

Suppose $f : [a, b]$ is integrable and for all partitions P

$$L(f, P) \leq A \leq U(f, P)$$

for some $A \in \mathbb{R}$, then $\int_a^b f = A$.

6.13.1.1. Proof

Since $L(f, P) \leq A$ for all P,

$$\underline{\int_a^b} \leq A$$

Similarly,

$$A \leq \overline{\int_a^b}$$

and so

$$\underline{\int_a^b} f \leq A \leq \overline{\int_a^b} f$$

and

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

So

$$\int_a^b f = A$$

6.13.2. Proof

Let P be any partition of $[a, b]$. We will show that

$$L(f, P) \leq F(b) - F(a) \leq U(f, P)$$

Since F' is continuous on (a, b) and bounded, F' is integrable and does not depend on $F'(a)$ or $F'(b)$.

For each $x_i \in P$, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i)$$

by the mean value theorem.

Therefore, $m_{i(x_i - x_{i-1})} \leq F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$ and $M_{i(x_i - x_{i-1})} \geq F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$

Then

$$U(F', P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(b) - F(a)$$

and similarly

$$L(F', P) \leq F(b) - F(a)$$

and therefore

$$\int_a^b F' = F(b) - F(a)$$

6.13.3. Results

Let $r > 0$.

$$\int_a^b x^r \, \mathrm{d}x = \frac{b^{r+1} - a^{r+1}}{r + 1}$$

Since $F(x) = x^{r+1}/(r + 1)$ and $F'(x) = x^r$, and the fundamental theorem of calculus.

But this is not always helpful. Finding the antiderivative of $\frac{1}{1+x^4}$ is very hard.

But this is possible by the additivity of the integral:

$$f(x) = \begin{cases} 0 & x \leq 2 \\ 4 & x > 2 \end{cases}$$

$$\int_0^6 f = \int_0^2 f + \int_2^6 f = 0 + 4 \cdot 6 - 4 \cdot 2 = 16$$

But this function is impossible to do with this method

$$f(x) = \begin{cases} 0 \text{ if } x \notin \mathbb{Q} \\ \frac{1}{n} \text{ if } x = \frac{m}{n} \in \mathbb{Q} \text{ and } \gcd(m, n) = 1 \end{cases}$$

The integral is 0 over any set (to be proven later).

6.14. The Mean Value Theorem for Integrals

If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f$$

6.14.1. Proof

By the extreme value theorem, there exists $m, M \in [a, b]$ such that

$$f(M) = \max_{x \in [a,b]} f(x)$$

and

$$f(m) = \min_{x \in [a,b]} f(x)$$

Then, by making a very bad partition $P = [a, b]$:

$$f(m)(b-a) \leq \int_a^b f \leq f(M)(b-a)$$

And therefore

$$f(m) \leq \frac{1}{b-a} \left(\int_a^b f \right) \leq f(M)$$

By the IVT, there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f$$

6.15. Integrals produce continuous functions

Suppose $f : [a, b] \to \mathbb{R}$ is integrable. Then, $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f$$

and $F(a) = 0$, is continuous.

6.15.1. Proof

Since f is integrable $\exists M > 0$ such that, for all $x \in [a, b]$, $-M \leq f(x) \leq M$.

Let $u, v \in [a, b]$ with $u < v$. Then

$$\begin{aligned} F(v) &= \int_a^v f \\ &= \int_a^u f + \int_u^v f \\ &= F(u) + \int_u^v f \end{aligned}$$

And then

$$F(v) - F(u) = \int_u^v f$$

By the MVT for integrals:

$$-M(v-u) \leq \int_u^v \leq M(v-u)$$

Thus

$$\left| \int_i^v f \right| \leq M|v-u|$$
$$|F(v) - F(u)| \leq M|v-u|$$

The same holds for $v \leq u$.

F is Lipschitz and, therefore, continuous.

6.16. Second Fundamental Theorem of Calculus

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f$$

and $F(a) = 0$ is differentiable, and $F'(x) = f(x)$.

6.16.1. Proof

Fix $x_0 \in (a, b)$. We will show that F is differentiable at x_0.

Let $x \in (a, b)$ such that $x \neq x_0$. If $x < x_0$:

$$F(x_0) = F(x) + \int_x^{x_0} f$$

If $x > x_0$:

$$F(x) = F(x_0) + \int_{x_0}^{x} f$$

Case 1:

$$F(x_0) - F(x) = \int_{x}^{x_0} f$$

By the MVT for integrals, there exists $c(x) \in (x, x_0)$ such that

$$f(c(x)) = \frac{1}{x_0 - x} \int_{x}^{x_0} f = \frac{F(x_0) - F(x)}{x_0 - x}$$

The same is true in case 2.

Since $c(x) \in (x, x_0)$ or $c(x) \in (x_0, x)$, $\lim\limits_{x \to x_0} c(x) = x_0$,

So $\lim\limits_{x \to x_0} f(c(x_0)) = f(x_0)$, and thus $\lim\limits_{x \to x_0} \frac{F(x_0) - F(x)}{x_0 - x} = f(x_0)$.

Therefore $F'(x_0) = f(x_0)$.

6.17. Backwards bounds derivative

For $x < b$

$$\left(\int_{x}^{b} f\right)' = -f(x)$$

6.17.1. Proof

$$0 = \left(\int_{a}^{b} f\right)' = \left(\int_{a}^{x} f + \int_{x}^{b} f\right)'$$

We conclude that

$$\left(\int_{x}^{b} f\right)' = -\left(\int_{a}^{x} f\right)' = -f(x)$$

6.18. Backwards Bounds

If f is integrable or $[a, b]$, then

$$\int_{b}^{a} f = -\int_{a}^{b} f$$

and $\int_{c}^{c} f = 0$.

6.19. Other functions in the top bound

If $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable and $f : [a, b] \to \mathbb{R}$ is continuous, then

$$F(x) = \int_a^{\varphi(x)} f$$

is differentiable as long as $\varphi(x) \in [a, b]$ and moreover

$$F'(x) = f(\varphi(x))\varphi'(x)$$

6.19.1. Proof

Let

$$G(x) = \int_a^x f$$

Then

$$F(x) = G(\varphi(x))$$

By the chain rule

$$F'(x) = G'(\varphi(x))\varphi'(x)$$

By the second FTC:

$$F'(x) = f(\varphi(x))\varphi'(x)$$

6.20. Natural Log

For $x > 0$:

$$\log(x) = \int_1^x \frac{1}{t} \, \mathrm{d}t$$

6.20.1. Properties

We know that this function is continuous, differentiable and strictly increasing.

1. $\forall a, b > 0$, $\log(ab) = \log(a) + \log(b)$
2. $\forall x > 0, r \in \mathbb{Q}, \log(x^r) = r \log(x)$
3. $\log : (0, \infty) \to \mathbb{R}$ is surjective.

6.20.1.1. Proof

First:

Fix $a > 0$. Define $h : (0, \infty) \to \mathbb{R}$ as $h(x) = \log(ax) - \log(a) - \log(x)$.

$$h(1) = \log(a) - \log(a) - \log(1) = -\log(1) = 0$$

$$\begin{aligned} h'(x) &= a \log'(ax) - 0 - \log'(x) \\ &= a \frac{1}{ax} - \frac{1}{x} \\ &= \frac{1}{x} - \frac{1}{x} \\ &= 0 \end{aligned}$$

Therefore $h'(x) = 0$ and so $h(x) = \mathbb{c}$, and $h(1) = 0$ and therefore $h(x) = 0$.

Second:
Let $h(x) = \log(x^r) - r\log(x)$.

$$h(1) = \log(1) - r\log(1) = 0$$

$$\begin{aligned} h'(x) &= rx^{r-1}\frac{1}{x^r} - r\frac{1}{x} \\ &= \frac{r}{x} - \frac{r}{x} \\ &= 0 \end{aligned}$$

Third:
Note $0 = \log(1) = \log\left(x\frac{1}{x}\right) = \log(x) + \log\left(\frac{1}{x}\right)$ and therefore $\log(x) = -\log\left(\frac{1}{x}\right)$.

So it is sufficient to show $\log : (1, \infty) \to (0, \infty)$ is surjective.

Let $c > 0$. Choose $n \in \mathbb{N}$ sufficiently large so that $n\log(2) > c$.

Then $\log(2^n) = n\log(2) > c$. Since $F(1) = 0$, and $c > 0$, there exists $x \in (1, 2^n)$ such that $\log(x) = c$.

6.21. Define e^x

$\log(x)$ is bijective from $(0, \infty)$ to $\mathbb{R}$.[5]

Therefore, there exists $g(x) : \mathbb{R} \to (0, \infty)$ such that $\forall x \in \mathbb{R}, \log(g(x)) = x$ and $\forall x > 0, g(\log(x)) = x$

Furthermore:

$$g'(x) = \frac{1}{\log'(g(x))} = \frac{1}{\frac{1}{g(x)}} = g(x)$$

Previously, we have defined a^x for $x \in \mathbb{Q}$.

But we know $\log(a^x) = x\log(a)$. Let us define this as a fact for all $\mathbb{R}$.

Then, $a^x = g(x\log(a))$.

There exists a unique number e such that $\log(e) = 1$.

Let $e^x = g(x\log(e)) = g(x)$.

6.22. e^x is unique

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable such that $f' = f$. Then, $f(x) = ce^x$ for some $c \in \mathbb{R}$.

6.22.1. Proof

Suppose $g' = g$. Define $h(x) = \frac{g(x)}{g(0)e^x}$

Then

[5] It is injective because it is strictly increasing.

$$h'(x) = \frac{g'(x)g(0)e^x - g(x)g(0)e^x}{\left(g(0)e^x\right)^2} = 0$$

$h(0) = 1$ so $h(x) = 1$ for all x.

Therefore $g(x) = g(0)e^x$.

6.23. Integration By Parts

Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b) with bounded derivatives. Then

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

6.23.1. Proof

By the product rule:

$$(fg)' = f'g + g'f$$

Thus:

$$\int_a^b (fg)' = f(b)f(b) - f(a)g(a) = \int_a^b (f'g + g'f)$$

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

6.24. Integration by substitution

Let $f : [a, b] \to \mathbb{R}$ be continuous and $g : [c, d] \to [a, b]$ be continuous and differentiable on (c, d) with a bounded derivative.

Then

$$\int_c^d f(g(x))g'(x)\,\mathrm{d}x = \int_{g(c)}^{g(d)} f(x)\,\mathrm{d}x$$

6.24.1. Proof

Let

$$H(x) = \int_c^x f(g(t))g'(t)\,\mathrm{d}t - \int_{g(c)}^{g(d)} f(t)\,\mathrm{d}t$$

$H(c) = 0$ by definition of integral from point to point.

$$H'(x) = f(g(x))g'(x) - f(g(x))g'(x) = 0$$

Therefore $H(x) = 0$, and so they are equal.

6.25. Darboux Sum Convergence

Let $f : [a, b] \to \mathbb{R}$ be bounded. Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the following are equivalent:

1. f is integrable

2. For all sequences of partitions with $\{P_n\}$ such that $\{\text{gap } P_n\} \to 0$:

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$$

6.25.1. Lemma

Suppose $f : [a, b] \to \mathbb{R}$ is bounded with $-M \leq f(x) \leq M$. Let P be a partition with k subintervals, and let P^* be any other partition. Then

$$U(f, P^*) \leq U(f, P) + kM \text{ gap } P^*$$
$$L(f, P^*) \geq L(f, P) - kM \text{ gap } P^*$$

6.25.2. Proof

Let $P^* = \{x_0, x_1, x_2, ..., x_n\}$. Let $P = \{z_0, z_1, ..., z_k\}$.

Let $C = \{i \in \{1, ..., n\} \,|(x_i, x_{i-1}) \cap P \neq \varnothing\}$.

$i \in C$ if (x_{i-1}, x_i) contains at least one z_j.

$$\sum_{i\in C} M_i(x_i - x_{i-1}) \leq kM \text{ gap } P^*$$

Let $P' = P \cup P^*$. Then:

$$\sum_{i\notin C} M_i(x_i - x_{i-1}) \leq U(f, P')$$

The above is incorrect for functions with negative regions across an interval where z is. By the refinement theorem:

$$\sum_{i\notin C} M_i(x_i - x_{i-1}) \leq U(f, P') \leq U(f, P)$$

Therefore

$$\begin{aligned} U(f, P^*) &= \sum_{i=1}^{n} M_i(x_i, x_{i-1}) \\ &= \sum_{i\in C}^{n} M_i(x_i, x_{i-1}) + \sum_{i\notin C}^{n} M_i(x_i, x_{i-1}) \\ &\leq kM \text{ gap } P^* + U(f, P) \\ &\leq U(f, P) + kM \text{ gap } P^* \end{aligned}$$

A similar argument holds for lower sums.

6.25.3. Proof

If for all sequences of partitions with $\{P_n\}$ such that $\{\text{gap } P_n\} \to 0$, $\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$, then there exists a sequence, and therefore by the Archimedes-Riemann theorem, f is integrable.

The other way:
Let be f integrable. Let $\varepsilon > 0$. Let $\{P_n\}$ be any sequence such that $\{\text{gap } P_n\} \to 0$.

We will show there exists an N such that if $n \geq N$ then:

$$|U(f, P_n) - L(f, P_n)| < \varepsilon$$

By the Archimedes-Riemann theorem, there exists a partition such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2}$$

Name the size of that partition k.

Let N be large enough that

$$2kM \text{ gap } P_n > \frac{\varepsilon}{2}$$

Therefore $U(f, P_n) - L(f, P_n) \leq U(f, P) + kM \text{ gap } P_n - L(f, P) + kM \text{ gap } P_n$.

Furthermore:

$$U(f, P) + kM \text{ gap } P_n - L(f, P) + kM \text{ gap } P_n < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

And therefore:

$$U(f, P_n) - L(f, P_n) < \varepsilon$$

7. Taylor Series

7.1. Contact Order

For $f, g : I \to \mathbb{R}$ have contact order 0 at $x_0 \in I$ if $g(x_0) = f(x_0)$.

Furthermore, if f and g are n-times differentiable, then they have contact order n if for all $k \in 1, ..., n$, $f^{(k)}(x_0) = g^{(k)}(x_0)$ and f and g have contact order 0.

7.1.1. Example

$f(x) = \sqrt{2 - x^2}$, $g(x) = e^{2-x}$. At $x_0 = 1$, these have a contact order of 1, but not a contact order of 2.

7.1.2. Example

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[(x - x_0)^n\right]\Big|_{x=x_0} = \begin{cases} 0 & k \neq n \\ n! & k = n \end{cases}$$

7.2. Taylor Polynomial

Let $f : I \to \mathbb{R}$ be n-times differentiable and let $x_0 \in I$. There exists a unique polynomial, p_n, of degree at most n that has contact order n with f at x_0. Moreover,

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$p_n(x)$ is the nth Taylor polynomial of f at x_0.

7.2.1. Proof of existence

By the previous example, and linearity of the the differential operator, $\forall k \in \{0, ..., n\}$:

$$p_n^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!} \times k! = f^{(k)}(x_0)$$

Therefore p_n has contact order n.

Suppose $q(x)$ is a polynomial of order at most n that has contact order n with f at x_0.

Because it is contact order n, $\forall k \in \{0, ..., n\}$:

$$q^{(k)}(x_0) = f^{(k)}(x_0)$$

This covers all terms of the sequence because each of these terms are linearly independent by creating an triangular matrix.

$$q(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n$$

By the previous example, and linearity of the the differential operator:

$$c_k = \frac{f^{(k)}(x_0)}{k!} \to q(x) = p_n(x)$$

7.2.2. Example

Find the nth Taylor polynomial for e^x at $x_0 = 0$.

$$
\begin{aligned}
f(x) = e^x &\to f(0) = 1 \\
f'(x) = e^x &\to f'(0) = 1 \\
&\vdots \\
f^{(n)}(x) = e^x &\to f^{(n)}(0) = 1
\end{aligned}
$$

And therefore

$$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n$$

7.2.3. Example

Find the nth Taylor polynomial for $f(x) = \log(x+1)$ at $x_0 = 0$.

$$
\begin{aligned}
f'(x) = \frac{1}{x+1} &\to f'(0) = 1 \\
f''(x) = -\frac{1}{(x+1)^2} &\to f''(0) = -1 \\
&\vdots \\
f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(x+1)^k} &\to f^{(k)}(0) = (-1)^{k+1}(k-1)!
\end{aligned}
$$

And therefore

$$p_n(x) = 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + \frac{(-1)^{n+1}}{n}x^n$$

7.2.4. Idea

If n is large, $p_n(x) \approx f(x)$ for $x \approx x_0$.

7.3. Lagrange Remainder Theorem

Let $f : I \to \mathbb{R}$ be $n+1$ times differentiable. Let $x_0 \in I$, let $p_n(x)$ be the nth taylor polynomial. Then, for all $x_0 \in I, x \in I$ such that $x \neq x_0$, there exists c between x and x_0 such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

7.3.1. Proof

Let $R(x) = f(x) - p_n(x)$,

Since the taylor polynomial has contact order n with f at x_0, $R(x_0) = R'(x_0) = \dots = R^{(n)}(x_0) = 0$.

By Section 5.19, there exists a c between x_0 and x such that:

$$R(x) = \frac{R^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Furthermore, since $p_n^{(n+1)} = 0$, $R^{(n+1)} = f^{(n+1)}$. Thus:

$$R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$\therefore \square$

7.4. e is irrational

e is irrational

7.4.1. Proof

First, we show, $e < 4$.

e is the unique number such that $\log(e) = 1$. Therefore, $e < 4$ iff $\log(4) > 1$, since $\log$ is strictly monotone increasing.

$$\log(4) = \int_1^4 \frac{1}{t} \,\mathrm{d}t \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$$

Therefore $e < 4$.

Since $e < 4$, $e^x \leq 4^x \forall x \geq 0$. Let $f(x) = e^x$. Let $n \in \mathbb{N}$. Then, centered at $x_0 = 0$ $p_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$.

Suppose e is rational. Then $e = \frac{a}{b}$ for some integers a, b. Assume $n \geq b$ and $n \geq 4$.

Then, $e - p_n(1) = \frac{a}{b} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n!}\right)$.

Furthermore, by Section 7.3, there exists $c \in (0, 1)$ such that

$$e - p_n(1) = \frac{e^c}{(n+1)!}(1)^{n+1} = \frac{e^c}{(n+1)!}$$

Therefore,

$$\frac{a}{b} - \left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}\right) = \left(\frac{a}{b}\right)^c \frac{1}{(n+1)!} \leq \frac{4^c}{(n+1)!} < \frac{4}{(n+1)!}$$
$$n!\frac{a}{b} - n!\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}\right) < \frac{4}{n+1} < 1$$

But $n!\frac{a}{b} - n!\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}\right) \in \mathbb{Z}$ since $n!\frac{a}{b} \in \mathbb{Z}$ since $n \geq b$ and $n!\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}\right) \in \mathbb{Z}$. Furthermore, since $\frac{e^c}{(n+1)!} > 0$, $e - p_n(1) > 0$.

Therefore we have found an integer between zero and one. But this is a contradiction, since there is no integer between zero and one.

Therefore the assumption that $e = \frac{a}{b}$ is false, and therefore $e \notin \mathbb{Q}$.

7.5. Euler Gamma

Let $a_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \log(n+1)$. Then $\{a_n\}$ is monotone increasing and bounded, and thus there exists $\gamma > 0$ such that $\{a_n\} \to \gamma$.

7.5.1. Proof

We will show

$$0 < x - \log(1+x) < \frac{x^2}{2}$$

for all $x > 0$.

Firstly, let $f(x) = x - \log(1+x)$. Then,

$$f'(x) = 1 - \frac{1}{1+x}$$

Since $\frac{1}{1+x} > 0$, $f'(x) > 0$, and so f is strictly monotone increasing. $f(0) = 0$, so for $x > 0$, $f(x) > 0$.

Let $g(x) = x^2 - x + \log(x+1)$.

$$g'(x) = x - 1 + \frac{1}{1+x}$$

$$g''(x) = 1 - \frac{1}{(1+x)^2} > 0$$

Since $g'(0) = 0$ and $g'(x)$ is strictly monotone increasing, $g'(x) > 0$.

Since $g(0) = 0$, for all $x > 0$, $g(x) > 0$.

Let $k \in \mathbb{N}$ and let $x = \frac{1}{k}$. Then,

$$0 < \frac{1}{k} - \log\left(1 + \frac{1}{k}\right) \leq \frac{1}{2k^2}$$

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{n+1} - \log(n+2) + \log(n+1) \\ &= \frac{1}{n+1} - \log\left(\frac{n+1+1}{n+1}\right) \\ &= \frac{1}{n+1} - \log\left(1 + \frac{1}{n+1}\right) \end{aligned}$$

for $k = n + 1$

$$= \frac{1}{k} - \log\left(1 + \frac{1}{k}\right)$$

Therefore

$$0 < a_{n+1} - a_n \leq \frac{1}{2k^2}$$

$$0 < a_{n+1} - a_n \leq \frac{1}{2(n+1)^2}$$

.

Thus, $\{a_n\}$ is monotone increasing.

7.6. Series

If the limit exists:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k$$

If the limit does not exist, we say that the infinite sum diverges.

7.7. Taylor Series

A sequence of Taylor polynomials $\{p_n\}$ for a function f at a point x_0 converges to f at x if $\lim_{n\to\infty} f(x) - p_n(x) = 0$.

If $p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$, then a Taylor series is:

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

7.8. A time a function is equal to its taylor series

Suppose $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose $r, M > 0$ and $x_0 \in I$ such that $\forall x \in [x_0 - r, x_0 + r]$, $\left|f^{(n)}(x)\right| \leq M^n$.

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

7.8.1. Lemma

Let $c > 0$. $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

7.8.1.1. Proof

Let $k \in \mathbb{N}$, $k \geq 2c$. If $n \geq k$,

$$\begin{aligned}
&\frac{c^n}{n!} \\
&= \left(\frac{c}{1} \cdot \frac{c}{2} \cdots \frac{c}{k}\right)\left(\frac{c}{k+1} \cdots \frac{c}{n}\right) \\
&\leq c^k \left(\frac{1}{2}\right)^{n-k} \\
&= c^k \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^k} \\
&= \frac{(2c)^k}{2^n}
\end{aligned}$$

Therefore

$$0 \leq \frac{c^n}{n!} \leq \frac{(2c)^k}{2^n}$$

Since $(2c)^k$ is constant in n and $\left\{\frac{1}{2^n}\right\} \to 0$, so therefore

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

7.8.2. Proof

By the Section 7.3, $\forall x \in [x_0 - r, x_0 + r]$, there exists $c \in [x_0 - r, x_0 + r]$ such that

$$|f(x) - p_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \right|$$

Then

$$\left| \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \right| \leq \frac{(M^{n+1} r^{n+1})}{(n+1)!} = \frac{c^{n+1}}{(n+1)!}$$

Let $\varepsilon > 0$. There $\exists N$ such that if $n \geq N$, then $\frac{c^{n+1}}{(n+1)!} \leq \varepsilon$ by the lemma.

Thus

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right| \leq \frac{c^{n+1}}{(n+1)!} < \varepsilon$$

and therefore

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x) \end{aligned}$$

7.8.3. e^x is analytic

$$\forall x \in \mathbb{R}, e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

7.8.3.1. Proof

Let $x_0 = 0$ and $r = |x|$. Let $f(x) = e^x$.

$f^{(n)}(x) = e^x$.

Then let $M = e^r$.

$$\left| f^{(n)}(x) \right| = |e^x| \leq e^r = M$$

Thus $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

7.9. Real-Analytic

$f : I \to \mathbb{R}$ is real-analytic in I if for all $x, x_0 \in I$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

7.9.1. Example

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not real-analytic, but is infinitely differentiable.

$$f^{(n)}(x) = \begin{cases} q_n\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Where q_n is some polynomial. But the derivatives are zero every time, so the taylor series converges to zero.

8. Sequences of Functions

8.1. Cauchy Sequences

A sequence $\{a_n\}$ is Cauchy if for all $\varepsilon > 0$, there exists N such that for all $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.

8.2. Sequences converge iff they are Cauchy

A sequence converges iff it is Cauchy.

8.2.1. Lemma

All Cauchy sequences are bounded.

8.2.1.1. Proof

Let $\{a_n\}$ be Cauchy. Let $\varepsilon = 1$. Then, there exists N such that $n, m \geq N$, $|a_n - a_m| < 1$.

In particular, $|a_n - a_N| < 1$.

$$|a_n| \leq \max\{|a_N| + 1, |a_1|, |a_2|, |a_3|, ..., |a_{N-1}|\}$$

8.2.2. Proof

$\leftarrow$ Suppose $\{a_n\}$ is Cauchy. Then, $\{a_n\}$ is bounded and a subsequence $\left\{a_{n_k}\right\}$ converging to a exists.

Fix $\varepsilon > 0$. Then, there exists N_1 such that if $k \geq N_1$, then $\left|a_{n_k} - a\right| < \frac{\varepsilon}{2}$. Also, there exists N_2 such that if $n, n_k \geq N_2$, then, $\left|a_n - a_{n_k}\right| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$,

$$|a_n - a| \leq \left|a_n - a_{n_k}\right| + \left|a_{n_k} - a\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

8.3. Convergent Series must go to zero

If $\sum_{k=1}^{\infty} a_k$ converges, $\{a_k\} \to 0$.

8.4. Prop

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \text{ if } |r| < 1$$

8.5. Monotone Convergence Theorem For Series

Suppose $\{a_k\}$ is a sequence of nonnegative numbers. Then $\sum_{k=1}^{\infty} a_k$ converges iff $\exists M > 0$ such that

$$\forall n, a_1 + a_2 + a_3 + \cdots + a_n < M$$

8.6. Comparison Test

Suppose $\{a_k\}, \{b_k\}$ are nonnegative such that $0 \leq a_k \leq b_k$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$

8.7. Integral Test

If $\{a_k\}$ is nonnegative and $f : [1, \infty) \to \mathbb{R}$ continuous and decreasing such that $f(k) = a_k$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ converges iff $\left\{\int_1^n f\right\}$ is bounded.

8.8. p-test

All sums of the the form below converge:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad p > 1$$

8.8.1. Proof

Let $f(x) = \frac{1}{x^p}$.

$$\int_1^n \frac{1}{x^p} \, \mathrm{d}x = \frac{x^{1-p}}{1-p}|_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

This converges, and therefore the above converges.

8.9. Alternating Series Test

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where $a_k \geq 0$, converges iff $\{a_k\} \to 0$.

8.10. Absolute Convergence implies convergence

$\sum_{k=1}^{\infty} a_k$ converges if $\sum_{k=1}^{\infty} |a_k|$ converges.

8.11. Ratio Test

Suppose $\lim\limits_{k \to \infty} \frac{a_{k+1}}{a_k} = \ell$. Then

1. $\sum_{k=1}^{\infty} a_k$ converges if $\ell < 1$.
2. $\sum_{k=1}^{\infty} a_k$ diverges if $\ell > 1$.

8.12. Riemann Rearrangement Theorem

Suppose $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ does not.

Then for all $x \in \mathbb{R}$, there exists a rearrangement of $\{a_n\}$ to a sequence $\{b_n\}$ such that

$$\sum_{n=1}^{\infty} b_n = x$$

8.13. Pointwise Convergence

$\{f_n\} \to f$ converges to f pointwise if $\forall x \in I$, $\lim\limits_{n \to \infty} f_{n(x)} = f(x)$.

8.13.1. Example

Let $f_n = x^n$, for $f_n : [0, 1] \to \mathbb{R}$.

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

8.13.2. Example

$f_n(x) = e^{-nx^2}$, for $f_n : \mathbb{R} \to \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 1 \end{cases}$$

8.13.3. Example

$$A = \mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, ...\}$$

Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, q_2, q_3, ..., q_n\} \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

8.13.4. Example

$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & x = 0 \vee \frac{1}{n} \leq x \end{cases}$$

$$\int_0^1 f_n = 1$$

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

Therefore,

$$\lim_{n\to\infty} \int_0^1 f_n = 1$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

8.13.5. Example

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$f_n : \mathbb{R} \to \mathbb{R}$

and then $\lim_{n\to\infty} f_n(x) = e^x$

8.14. Uniform Convergence

Let $f_n : I \to \mathbb{R}$ for all n. $\{f_n\}$ converges uniformly to $f : I \to \mathbb{R}$ if $\forall \varepsilon, \exists N, \forall n \geq N, \sup_{x\in I} |f_n(x) - f(x)| < \varepsilon$.

8.15. Uniform Cauchy

Let $f_n : I \to \mathbb{R}$ for all n. $\{f_n\}$ is uniformly Cauchy if $\forall \varepsilon > 0, \exists N, \forall n, m \geq N, \sup_{x\in I} |f_n(x) - f_m(x)| < \varepsilon$

8.16. Weierstrass Uniform Convergence Criterion

Let $f_n : I \to \mathbb{R}$. Then $\{f_n\}$ is uniformly convergent of $f : I \to \mathbb{R}$ iff $\{f_n\}$ is uniformly Cauchy.

8.16.1. Proof

$\to$ Suppose $\{f_n\} \to f$ uniformly.

Fix $\varepsilon > 0$. There exists N such that $\forall n \geq N$

$$\sup_{x \in I} |f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

Then, if $n, m \geq N$

$$\sup_{x \in I} |f_n(x) - f_m(x)| \leq \sup_{x \in I} (|f_n(x) - f(x)| + |f(x) - f_m(x)|) \leq \sup_{x \in I} |f_n(X) - f(x)| + \sup_{x \in I} |f(x) - f_m(x)|$$

$\leftarrow$ Suppose $\{f_n\}$ is uniformly Cauchy.

Fix $\varepsilon > 0$. Then there exists N such that if $n, m \geq N$

$$\sup_{x \in I} |f_n(x) - f_m(x)| < \varepsilon$$

Therefore, for all x, $\{f_n(x)\}$ is Cauchy. Thus, there exists a y_x such that $\{f_n(x)\} \to y_x$. Define $f(x) = y_x$.

If $n, m \geq N$ for all x,

$$-\frac{\varepsilon}{2} \leq f_n(x) - f_n(x) \leq \frac{\varepsilon}{2}$$

WLOG, $m > n$, $m = n + k$ for some k.

$$f_n(x) - \frac{\varepsilon}{2} \leq f_{n+k}(x) \leq f_n(x) + \frac{\varepsilon}{2}$$

$$\lim_{k \to \infty} f_n(x) \pm \frac{\varepsilon}{2} = f_n(x) \pm \frac{\varepsilon}{2}$$
$$\lim_{k \to \infty} f_{n+k}(x) = f(x)$$

Then

$$f_n(x) - \frac{\varepsilon}{2} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{2}$$

and

$$\sup_{x \in I} |f(x) - f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

8.17. Cantor Set

Let $I_0 = [0, 1]$. Let $I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. $I_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Think of this like repeatedly taking out the center.3/9

For I_n, this is the union of 2^n closed intervals of length $1/3^n$.

$$\mathcal{C} = \bigcap_{n=0}^{\infty} I_n$$

$\mathcal{C}$ is the Cantor set.

$$0 \in \mathcal{C}, \frac{1}{3} \in \mathcal{C}, \frac{2}{3} \in \mathcal{C}, 1 \in \mathcal{C}.$$

Define

$$X = \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{c_n}{3^n}, c_n = 0 \vee 2 \forall n \right\}$$

We claim $X = \mathcal{C}$.

Furthermore, $\mathcal{C}$ is uncountably infinite. If $x \in \mathcal{C}$, $x = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$, $c_n = 0 \vee 2$. Suppose $\mathcal{C}$ is countably infinite. Then, there exists a bijection $f : \mathbb{N} \to \mathcal{C}$.

$$\begin{aligned} f(1) &= \sum_{n=1}^{\infty} \frac{c_n^1}{3^n} \\ f(2) &= \sum_{n=1}^{\infty} \frac{c_n^2}{3^n} \\ &\vdots \\ f(k) &= \sum_{n=1}^{\infty} \frac{c_n^k}{3^n} \end{aligned}$$

Let

$$c_n^\star = \begin{cases} 0 & \text{if } c_n^n = 2 \\ 2 & \text{if } c_n^n = 0 \end{cases}$$

Let

$$x^\star = \sum_{n=1}^{\infty} \frac{c_n^\star}{3^n}$$

Note that $x^\star \in \mathcal{C}$ but it is also not equal to any of $f(1), f(2), \ldots$, which is a contradiction, and therefore $\mathcal{C}$ is uncountably infinite.

Let $\lambda(S)$ be the length of a set S.

- $\lambda(S) \geq 0$ forall $S \subset \mathbb{R}$.
- If $S \subset S'$, then $\lambda(S) \leq \lambda(S')$
- $\lambda([a,b]) = b - a$
- $S \cap S' = \varnothing$, then $\lambda(S \cup S') = \lambda(S) + \lambda(S')$

$$\lambda(\mathcal{C}) \leq \lambda(I_n) = \left(\frac{2}{3}\right)^n \to \lambda(\mathcal{C}) = 0$$

- $\mathcal{C}$ is uncountably infinite.

- It has length 0.
- It is a closed set.
- No isolated points
  - all points are limit points
- Totally disconnected
  - contains no open intervals

Let $f_0(x) = x$.

Let $f_n(0) = 0, f_n(1) = 1$. $f_{n'}(x) = 0$ if $x \notin I_n$.

Then, $\{f_n\}$ is uniformly Cauchy, and so $\{f_n\} \to f$ uniformly. Each f_n is continuous and therefore f is continuous. f is also called the Devil's staircase.

8.18. Uniform Convergence Preserves Continuity

Suppose $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuous functions converging uniformly to $f : I \to \mathbb{R}$. Then f is continuous.

8.18.1. Proof

Fix $\varepsilon > 0$. Then there exists N such that if $n \geq N$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Let $x, y \in I$. Then

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}$$
$$|f_N(y) - f(y)| < \frac{\varepsilon}{3}$$

Furthermore, f_N is continuous at x, so there exists $\delta > 0$ such that if $|x - y| < \delta$, there exists

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

Thus, if $|x - y| < \delta$:

$$\begin{aligned} & |f(x) - f(y)| \\ &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(X) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

And therefore f is continuous.

8.19. Distance Metric in the Space of Continuous Bounded Functions

The space of all continuous bounded functions on a set $A \subset \mathbb{R}$ is $C^0(A)$ and the distance between two functions $f, g \in C^0(A)$ is

$$d(f, g) = \sup_{x \in A} |f(x) - g(x)|$$

8.20. Uniform Convergence allows Limit-Integral Interchange

Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ is a sequence of integrable functions uniformly converging to $f : [a,b] \to \mathbb{R}$. Then f is integrable and

$$\int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n$$

8.20.1. Proof

If $h(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\underline{\int_a^b} h < \underline{\int_a^b} g$$
$$\overline{\int_a^b} h < \overline{\int_a^b} g$$

To show that f is integrable, we will show that for all $\varepsilon > 0$,

$$0 < \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$$

Fix $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{3(b-a)}$.

Then there exists N such that if $n \geq N$, for all $x \in [a,b]$:

$$f_n(x) - \varepsilon' \leq f(x) \leq f_n(x) + \varepsilon'$$

Thus

$$\underline{\int_a^b} (f_n - \varepsilon') \leq \underline{\int_a^b} f \leq \underline{\int_a^b} (f_n + \varepsilon')$$
$$\underline{\int_a^b} f_n - \frac{\varepsilon}{3} \leq \underline{\int_a^b} f \leq \underline{\int_a^b} f_n + \frac{\varepsilon}{3}$$

And similarly:

$$\overline{\int_a^b} f_n - \frac{\varepsilon}{3} \leq \overline{\int_a^b} f \leq \overline{\int_a^b} f_n + \frac{\varepsilon}{3}$$

Therefore:

$$\begin{aligned} 0 &\leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq \overline{\int_a^b} + \frac{\varepsilon}{3} - \underline{\int_a^n} f_n + \frac{\varepsilon}{3} \\ &\leq \frac{2\varepsilon}{3} \\ &< \varepsilon \end{aligned}$$

Therefore f is integrable. Furthermore

$$\left(\int_a^b f_n\right) - \frac{\varepsilon}{3} < \int_a^b < \left(\int_a^b f_n\right) - \frac{\varepsilon}{3}$$

$$\left|\int_a^b f - \int_a^b f_n\right| < \varepsilon$$

Thus, for all $\varepsilon > 0$, there exists N such that if $n \geq N$,

$$\left|\int_a^b f - \int_a^b f_n\right| < \varepsilon$$

and therefore:

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

8.21. Continuously Differentiable

A function $f : I \to \mathbb{R}$ is continuous differentiable if it is differentiable and f' is continuous.

8.22. Differentiability of Pointwise Limit Function with Uniform Derivative Convergence

Suppose $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions such that:

1. $\{f_n\} \to f$ pointwise in I.
2. $\{f_{n'}\} \to g$ uniformly in I.

Then, f is differentiable and $f' = g$.

8.22.1. Proof

Pick $x_0 \in I$. Then, for all n,

$$f_n(x) - f_n(x_0) = \int_{x_0}^x {f_n}'$$

Furthermore

$$\lim_{n\to\infty} f_n(x) - f_n(x_0) = f(x) - f(x_0)$$

Thus:

$$\begin{aligned} & f(x) - f(x_0) \\ = & \lim_{n\to\infty} f_n(x) - f_n(x_0) \\ = & \lim_{n\to\infty} \int_{x_0}^x f_{n'} \\ = & \int_{x_0}^x \lim_{n\to\infty} f_{n'} \end{aligned}$$

$$= \int_{x_0}^{x} g$$

9. Power Series

9.1. Power series

A power series is any series of the form

$$\sum_{k=0}^{\infty} c_k x^k$$

The domain of convergence of a power series is the set of all values of x such that the series converges.

Furthermore, 0 is always in that set.

9.1.1. Example

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Iff $-1 < x < 1$.

9.1.2. Example

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k+2}$$

Perform the ratio test:

$$\begin{aligned} &\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{k+3} \frac{k+2}{(-1)^k x^k} \right| \\ &= \lim_{k\to\infty} |x| \frac{k+2}{k+3} \\ &= \lim_{k\to\infty} |x| \end{aligned}$$

Therefore, the series converges $-1 < x < 1$

9.2. Analytic functions are continuous when uniformly convergent

Let

$$f_n(x) = \sum_{k=0}^{n} c_k x^k$$

If $\{f_n(x)\}$ converges uniformly to f defined as

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

Then f is continuous and integrable. Furthermore:

$$\int_a^b f = \sum_{k=0}^{\infty} \int_a^b c_k x^k$$

9.3. Relating Power and Taylor Series

Let $r > 0$ such that $(r, -r) \subset D$, where D is the domain of convergence of a power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = f(x)$$

then the power series is infinitely differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(c_k x^k\right)$$

In particular,

$$c_n = \frac{f^{(n)}(0)}{n!}$$

9.3.1. Lemma

Let A be a subset of the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$. If there exists $M > 0$ and $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ such that for all $x \in A$ and for all $k \in \mathbb{N}$, $\left|c_k x^k\right| \leq M\alpha^k$ then the power series converges uniformly in A.

9.3.1.1. Proof

Let $\varepsilon > 0$. Let N be large enough such that

$$\sum_{k=N}^{\infty} M\alpha^k < \varepsilon$$

This is possible since

$$\begin{aligned}
&\sum_{k=N}^{\infty} M\alpha^k \\
&= \sum_{k=0}^{\infty} M\alpha^k - \sum_{k=0}^{N-1} M\alpha^k \\
&= \frac{M}{1-\alpha} - \frac{M(1-\alpha^N)}{1-\alpha} \\
&= \frac{M\alpha^N}{1-\alpha}
\end{aligned}$$

And since $0 \leq \alpha < 1$, N can be made large enough.

Then $\forall x \in A$, if $n \geq N$:

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{n} c_k x^k\right| \\
&= \left|\sum_{k=n+1}^{\infty} c_k x^k\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=n+1}^{\infty} \left|c_k x^k\right| \\
&= \sum_{k=n+1}^{\infty} M\alpha^k \\
&= \sum_{k=N}^{\infty} M\alpha^k \\
&< \varepsilon
\end{aligned}
$$

Therefore it is uniformly convergent.

9.3.2. Lemma

If $0 < \alpha < \beta$, then there exists $c \geq 0$ such that $k\alpha^k \leq c\beta^k$ for all $k \in \mathbb{N}$.

9.3.3. Proposition

Let the power series

$$\sum_{k=0}^{\infty} c_k x^k$$

have domain of convergence D. Let $x_0 \in D$, then if $0 < r < |x_0|$, then $[-r, r] \subset D$, and also a subset of the domain of convergence of

$$\sum_{k=1}^{\infty} k c_k x^{k-1}$$

Moreover, both converge uniformly on $[-r, r]$

9.3.3.1. Proof

The terms of $\left\{\left|c_k x_0^k\right|\right\}$ are bounded, so there exists an M such that

$$\left|c_k x_0^k\right| \leq M$$

Let $\alpha = r/|x_0| < 1$. Then, $\forall x \in [-r, r]$:

$$
\begin{aligned}
&\left|c_k x^k\right| \\
&= \left|c_k \left(\frac{x}{x_0}\right)^k x_0^k\right| \\
&= \left|c_k x_0^k\right| \left|\frac{x}{x_0}\right|^k \\
&\leq M \left|\frac{r}{x_0}\right|^k \\
&= M\alpha^k
\end{aligned}
$$

By the previous lemma, the power series $\sum_{k=0}^{\infty} c_k x^k$ converges uniformly in $[-r, r]$

$$\begin{aligned}&\sum_{k=1}^{\infty} k c_k x^{k-1} \\ &= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k\end{aligned}$$

Observe that

$$\begin{aligned}&\left|(k+1) c_{k+1} x^k\right| \\ &\leq (k+1)\left|c_{k+1}\right| r^k \\ &\leq \frac{k+1}{r}\left|c_{k+1}\right| r^{k+1} \\ &\leq \frac{k+1}{r} M \alpha^{k+1} \\ &= \frac{k+1}{kr} k M \alpha^{k+1} \\ &\leq \frac{2M\alpha}{r} k \alpha^k\end{aligned}$$

By a lemma, letting $\alpha' = \frac{\alpha+1}{2}$, there exists c such that $k\alpha^k \leq c\alpha' k$ for all k.

Therefore

$$\leq \underbrace{\frac{2M\alpha c}{r}}_{M'} \alpha'^k$$

And so the following converges uniformly in $[-r, r]$:

$$\sum_{k=1}^{\infty} k c_k x^{k-1}$$

9.3.4. Proposition

If $\{c_n\}$ is a sequence of nonnegative integers such that $\sum_{k=0}^{\infty} c_k$ converges and $\{h_k(x)\}$ is a sequence of continuous function on a set $A \subset \mathbb{R}$ such that for all x, k, $|h_k(x)| < c_n$, then $\sum_{k=0}^{\infty} h_k(x)$ converges uniformly $f : A \to \mathbb{R}$ and f is continuous.

9.3.4.1. Proof

Let $\varepsilon > 0$. Let N be large enough such that

$$\sum_{k=N}^{\infty} c_k < \varepsilon$$

If $n \geq N$,

$$\left|\sum_{k=0}^{\infty} h_k(x) - \sum_{k=0}^{n} h_k(x)\right|$$

$$
\begin{aligned}
&= \left| \sum_{k=n+1}^{\infty} h_k(x) \right| \\
&< \sum_{k=n+1}^{\infty} |h_k(x)| \\
&< \sum_{k=n+1}^{\infty} c_k \\
&\leq \sum_{k=N}^{\infty} c_k \\
&< \varepsilon
\end{aligned}
$$

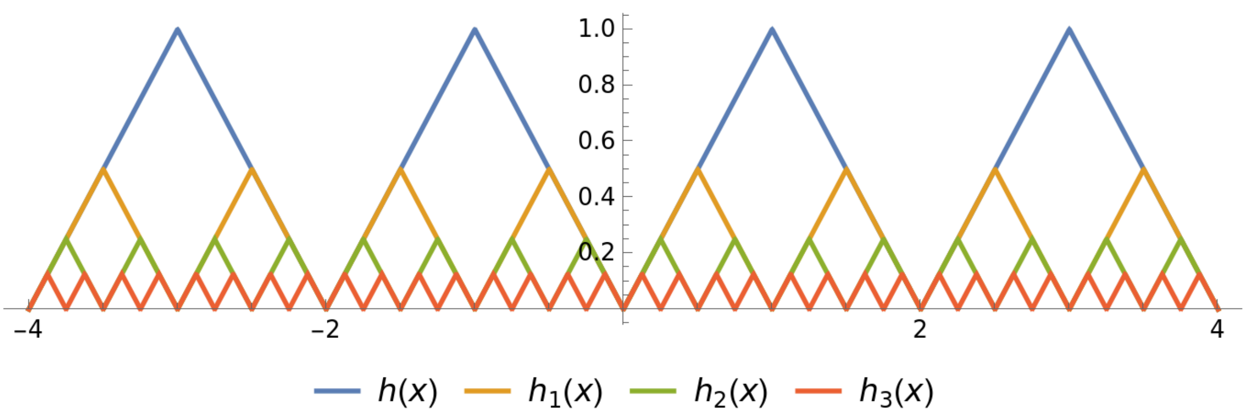
9.4. Function that is not differentiable anywhere

Let

$$
h(x) = \begin{cases} \vdots & \\ |x+2| & \text{if } -3 \leq x \leq -1 \\ |x| & \text{if } -1 \leq x < 1 \\ |x-2| & \text{if } 1 \leq x \leq 3 \\ \vdots & \end{cases}
$$

or $h(x) = 1 - |\operatorname{mod}(x, 2) - 1|$.

Define $h_k(x) = \frac{1}{2^k} h(2^k x)$



Define $f(x) = \sum_{k=0}^{\infty} h_k(x)$

f is continuous on $\mathbb{R}$ but not differentiable at any point.

9.4.1. Lemma

Let $x_0 \in \mathbb{R}$, then h_k is monotone on either $\left[x_0, x_0 + \frac{1}{2^{k+1}}\right]$ or $\left[x_0 - \frac{1}{2^{k+1}}, x_0\right]$.

9.4.2. Proof

$|h_k(x)| \leq \frac{1}{2^k}$ for all x, k and thus by proposition, f is continuous.

We need to, for any x_0, find a sequence $\{x_n\} \to x_0$ such that

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

Let $x_0 \in \mathbb{R}$. For all n, h_n is monotone on $\left[x_0, x_0 + \frac{1}{2^{n+1}}\right]$ or $\left[x_0 - \frac{1}{2^{n+1}}, x_0\right]$.

In the first case, let $x_n = x_0 + \frac{1}{2^{n+1}}$. In the second case, let $x_n = x_0 - \frac{1}{2^{n+1}}$.

Then h_n is monotone between x_0 and x_n.

$$\begin{aligned}
&\frac{f(x_n) - f(x_0)}{x_n - x_0} \\
&= \frac{\left[\sum_{k=0}^{\infty} h_k(x_n)\right] - \left[\sum_{k=0}^{\infty} h_k(x_0)\right]}{x_n - x_0} \\
&= \sum_{k=0}^{\infty} \frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}
\end{aligned}$$

If $0 \leq k \leq n$:

$$\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0} = \pm 1$$

Note: missing case for $n = k + 1$. This doesn't really change very much

If $k + 1 > n$:

$$\begin{aligned}
&h_k(x_n) - h_k(x_0) \\
&= h_k\left(x_0 \pm \frac{1}{2^{n+1}}\right) - h_k(x_0) \\
&= 0
\end{aligned}$$

Since, for all $i \in \mathbb{Z}, x \in \mathbb{R}$:

$$h_k(x) = h_k\left(x + \frac{i}{2^{k-1}}\right)$$

And

$$\begin{aligned}
\frac{1}{2^{n+1}} &= \frac{i}{2^{k-1}} \\
\frac{2^{k-1}}{2^{n+1}} &= i \\
2^{k-1-(n+1)} &= i \\
2^{k-n-2} &= i
\end{aligned}$$

Therefore:

$$\sum_{k=0}^{\infty} (h_k(x_n) - h_k(x_0))$$

$$= \sum_{k=0}^{n} \frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}$$

$$= \begin{cases} \text{even} & \text{if } n \text{ is odd} \\ \text{odd} & \text{if } n \text{ is even} \end{cases}$$

And since $\frac{f(x_n)-f(x_0)}{x_n-x_0}$ is a sequence that alternates between even and odd, the limit as $n \to \infty$ does not converge, and therefore the function is not differentiable.